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REHABILITATION OF GRADUATE WORK*

J. R. KLINE, University of Pennsylvania

Your commitee has asked me to discuss the subject of the Rehabilitation of Graduate Work. In this address I shall confine the discussion to the rehabilitation of graduate work in mathematics. Emphasis must, however, be placed upon the fact that the rebuilding of graduate work in *all fields of knowledge* is of vital concern to mathematicians and, in fact, to any group whose primary interest is in a particular science. Just as the human body is a unit and all parts must be strong and well-coordinated to insure a healthy being, so physical science and mathematics cannot flourish while the humanities and other academic fields are neglected, so far as research and graduate activity are concerned. If an overwhelming and disproportionate amount of the nation's superior talent is drawn into science, the result will be detrimental to the nation and to science. Plans for the recovery and development of scientific talent must be related to society's other needs for intellectual leadership. The problem of the rehabilitation of graduate work in the social sciences and the humanities is perhaps an even more difficult one than in mathematics, for full-scale training at the graduate level has been interrupted in those fields for a longer period than in science.

Let us review briefly the growth of graduate work in mathematics in the United States as shown by the number of Ph.D. degrees granted in various periods.

Period	Average Number of Ph.D.'s Granted Annually
1900–04	12
1905–09	18
1910–14	24
1915–19	24
1920–24	24
1925–29	45

It is interesting to note that during the period 1915–19, the same average was maintained as in the previous five years, in spite of the fact that the First World War occurred in this interval and that in the academic year 1918–19, there were only nine doctorates in mathematics. In the period 1920–24, the number did not increase because of the interruption of training of undergraduate students during the war; these men would have been the potential graduate students for the first half of the twenties. In the second half of the twenties, graduate work was increasing very rapidly; in 1929, there were sixty-three doctorates awarded in mathematics.

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* Address delivered at the Annual Meeting of the Mathematical Association of America on November 25, 1945.

For the remainder of the period which I am discussing, the figures are given for each year.

Year	No. of Ph.D.'s	Year	No. of Ph.D.'s
1930	78	1938	69*
1931	76	1939	104
1932	80	1940	89
1933	77	1941	104
1934	96	1942	76
1935	86	1943	51
1936	82	1944	39
1937	72*		

During the period 1940–44, the number of master's degrees granted were as follows:

Year	No. of Master's Degrees
1940	303
1941	175
1942	168
1943	112
1944	114

Observe the very rapid decrease in the number of doctorates and master's degrees since 1941. You will note that, in 1944, American universities granted less than forty per cent of the number of doctorates conferred in 1941. Figures are not available for 1945 but the number is probably smaller than in 1944.

Let us examine the reasons for this rapid decrease. The first of these is the fact that great numbers of graduate students and prospective graduate students, who were undergraduate majors in mathematics, entered the armed forces. Some of these were driven by social pressure to volunteer when it would have been to the best interests of the country for them to continue their scientific training. A much greater number were inducted under the operation of the Selective Service Act. Until July, 1942, mathematicians did not qualify for deferment and both graduate students and teachers of mathematics were classified I-A, with practically no opportunity for successful appeal. Finally, success crowned the efforts of those who argued that the country should conserve and increase its supply of mathematicians and in July, 1942, mathematicians were listed among those engaged in critical occupations. However, even then a graduate student was eligible for deferment only if he were also teaching at least fifteen hours per week. This left very little time for effective graduate work. Under the operation of this directive, deferment of graduate students and teachers of mathematics was not automatic. Cases were sometimes decided adversely by local boards which, while doing a magnificent piece of work in general, were not in possession of the expert knowledge necessary for the appraisal of highly specialized personnel.

* The drop in the number of Ph.D.'s in these years may be due, in part, to the decrease in undergraduate attendance in our colleges and universities during the depression.

However, there was a period, beginning in May, 1943, when Selective Service machinery was put on a satisfactory basis by the creation of the National Committee on Physicists and Mathematicians. This was again changed in November, 1943, by the abolition of this committee and the substitution of procedures involving certification by the United States Employment Service. This machinery was abolished in April, 1944, insofar as men under twenty-six were concerned and, consequently, almost all physically able men engaged in teaching and studying mathematics were inducted. As a result, graduate work in mathematics was carried on only by women students, men classified as IV-F, and men in war industry who carried on graduate work in the evening. The number of doctorates awarded since 1941 would have been still smaller, if there had not been a group of men and women who had completed the course work and had started on dissertations before being drawn into war work. Many of these persons were able to finish their dissertations while engaged in war activities.

Another reason for the decrease in the number of doctorates is that many persons entered war work in connection with various projects such as the Radiation Laboratory and Aberdeen Proving Ground, because they wished to use their mathematical training as fully as possible for the nation's war effort. They were not at all certain that their training would be used effectively by the armed forces and, consequently, they chose to enter fields where there was emphasis on mathematical and physical training. In many cases, men of this group are now being inducted because the agencies can no longer ask for their deferment. Unless remedial action can be taken as indicated later, these men will probably be in the service for a long time because they will have very low point scores and will be the last persons demobilized.

It must be borne in mind that many women who were potential graduate students also interrupted their work to enter various phases of war activity as computers and statisticians.

A discouraging feature of this situation is the fact that we cannot rely upon the undergraduate majors in science to furnish us with the new supply of graduate students. Undergraduate students, other than engineers and pre-medical students, had even less protection than graduate students and, consequently, the number of available male graduates from colleges and universities, with majors in mathematics, is very far below normal.

M. H. Trytten of the Office of Scientific Personnel has made an extensive study of the shortages in the various sciences, due to the cessation of training during the war. It is his conservative opinion that it will be 1955 before we will be producing the number of Ph.D.'s that we would have expected to produce each year, if the growth of graduate work had continued at the same rate as in the decade 1930–40. Thus we will have lost a large number of trained scientists during the period 1940 to 1955.

Let us examine the effect of this loss of scientists. In the first place, *it lowers the scientific competence of the nation*. I hold it as fundamental that one of the greatest sources of strength that any nation can have is that power which comes

to it through its *research* in fundamental science. Kill the impulse that leads scholars to devote their full efforts to the search for truth for the sake of truth alone, and you have destroyed the impetus for intellectual progress and one of the nation's richest heritages. Mathematics, that was developed merely as a purely intellectual investigation into the consequences of certain assumed hypotheses, has made tremendous contributions to all phases of our national war effort. Likewise, it has been the means by which other sciences have been able to make much more significant contributions.

In order that real progress may be made, scientific research must always continue as a great adventure into the unknown. I quote from the physicist, Rabi: "To set out a detailed program with merely practical goals for truly scientific research is like trying to make a map of a country no one has ever seen and the very existence of which is in grave doubt. Pure science cannot have any goal other than the appeasement of the human spirit of intellectual adventure." I hold that if progress in pure science is not maintained by training a future generation that will make up the deficit that now exists in our personnel, then truly we shall lose the peace, even though we have won the war.

At this point I should like to quote from a recent article by Vannevar Bush:

"Americans who sincerely want to share in securing and maintaining the peace of the world through a strong international organization have their work plainly before them. We must all judge our courses as good or bad according to how they will help or hinder the strength with which the nation goes forward, not as a possible combatant in a world of isolated, suspicious states but as one free people among other free peoples. This applies to the engineer in the fields, the scientist in the laboratory, the industrialist at his desk, the mechanic at the lathe—to all of us as individuals, and to all of us as groups. No individual, no group has any right to override the good of the nation as a whole. At this critical time, any individual who places a selfish interest above the good of his country, above its strength in a difficult world, should pause to think that by so doing he may be making it necessary for his children or grandchildren to fight in a desperate war.

"The realistic sense of moral responsibility which I stress here is basic. With it, we have the foundation on which the bulwarks of national strength can be built.

"I place high in the list of these the maintenance of vigorous research in fundamental science, with Federal financial aid for the support of research programs and for the education of future scientists, and with stress on the fact that fundamental research demands at the same time the highest degree of freedom and initiative for the individual."

In the second place, *these scientists are needed by colleges and universities* for their research staffs and for their teaching staffs. The amount of teaching of mathematics which must be done by our colleges and universities in the post-war years will be very great. Those of you who remember the period just after the First World War will recall the tremendous influx of students to our colleges in

the years 1919 and 1920. Even though graduate training had been interrupted for a much shorter period, the number of men who had completed the training up to the doctorate was inadequate to meet the demand. Many institutions of good standing were not able to maintain the requirement that an instructor hold the Ph.D. degree; indeed, many were unable to do so until about 1925 and 1926.

Let us look at the present situation. The figures to be given are based on a study made during the past summer of staffs of all colleges and universities. According to these statistics, there were 1,675 persons with the Ph.D. degree among the 4,600 teachers of mathematics who were employed by the colleges and universities of this country. The number 4,600 includes all members of mathematical teaching staffs of the rank of instructor and above. Included in the tabulation were colleges and universities, some professional and technological institutions, most junior colleges, and teachers colleges. Of the 4,600 teachers of mathematics, about 11% were on leave of absence.

It is interesting to analyze these figures still further. I have divided the institutions as follows: (1) Member institutions of the Association of American Universities; (2) Institutions approved by the A.A.U.; (3) Institutions not approved by the A.A.U. (*i.e.*, those institutions which, according to A.A.U. judgment, do not maintain curricula and standards which fit their students for graduate work).

(1) In the first group, member institutions of the Association of American Universities (33 institutions), there are 449 persons with the rank of instructor or higher. Of these, 355 (or approximately 80%) hold the Ph.D. degree. This percentage is surely lower than normal because there are a number of notable instances where institutions have been forced to appoint to instructorships persons without doctor's degrees, in order to meet their teaching demands. This is a policy which these institutions would not have countenanced in pre-war years.

(2) Institutions on the approved list of the A.A.U. In this list there are 266 colleges and universities, having 1,240 members of the mathematics departments. Of these 1,183 are of the rank of instructor or higher and 677 hold the Ph.D. degree; in other words, 57% of those who are instructors have attained the doctorate. In addition, there are 23 approved technological institutions, such as Georgia School of Technology, with 315 members of the mathematics departments, of whom 251 are of the rank of instructor or higher. Of these, 146, or 58%, hold the doctorate. Thus, in the institutions of each type in this group, there is approximately the same percentage of persons with the Ph.D. degree.

(3) Non-approved institutions. Slightly more than 2,500 of the teachers of mathematics in colleges and universities are in institutions not on the approved list of the A.A.U. About 500 persons with the Ph.D. degree are at institutions of this group. Thus in institutions of this group, about 20% of the mathematics faculties hold the Ph.D. degree.

We may expect that there will be a great demand from institutions in this third group for mathematicians with advanced training up to the doctorate, in order to improve their academic standing and thereby gain recognition and

approval of their students for graduate work. It is the general practice for most institutions to refuse admission to the graduate schools of students from non-approved institutions, unless these students qualify on the basis of the Graduate Record Examination. Prospective graduate students from this group are not, by any means, to be disregarded; from some of the small colleges we may secure excellent graduate students. In fact, this spring there came to my attention a candidate from a non-approved institution who made one of the best records on the Graduate Record Examination I have ever seen.

It should be borne in mind that the figures given above display a somewhat distorted picture with respect to the number of doctors on mathematical staffs. Many departments of mathematics in our universities and colleges have suffered losses by death and retirement during the war and have made no permanent solution of the problem of filling these vacancies but have made temporary appointments. Many professors who have reached the age of retirement have been requested to continue until the close of the war, but these men will wish to retire now.

Another factor that must be borne in mind is the fact that industry and governmental agencies are likely to employ men trained in graduate mathematics in increasing numbers, thus making still greater demands upon our supply of mathematicians.

To meet the demands of both the academic and the industrial world, we must first see that there is an adequate supply of superior men to teach and to supervise the research of those taking graduate work in mathematics. In this country, we have been fortunate that there have been no casualties among mathematicians in war service, except indirectly as the deaths of certain men may have been hastened by overwork occasioned by their devotion to the nation's war effort. In this respect, our situation is entirely different from that of Poland where probably two-thirds of the nation's mathematicians were murdered for political and racial reasons, were killed in battle, or died from causes directly attributable to the war. Hungary has also lost heavily.

However, this does not mean that all trained in mathematics are available to the universities and colleges. Many of them are still on leave from their permanent posts. Many will receive attractive offers to remain in or enter government service and industry. If university research and training is to keep pace with the growth of our economy, if able men are to remain in college research and others are to be attracted to it, we must meet the following problems.

(1) New sources of financial support must be found. Incomes in other professions—medicine, law, engineering, *etc.*—have increased steadily during the war. The cost of living has risen markedly. The personal income tax bears heavily on the middle income groups. In spite of these factors, most university salaries have been frozen at a level which was none too high even before the war. A retention of this scale will have an adverse effect on the recruitment and retention of college and university personnel.

(2) University teaching loads must not be excessive, if good men are to return and others are to be retained in academic life. In many cases, research men have been drawn into positions where all their time and energy was to be devoted to research in problems connected with the war. Many of these men will wish to return to teaching, for teaching and research are complementary functions. If there is too much teaching, especially of an elementary character, research is bound to suffer and many men will be attracted to the outside positions which offer full time for research as well as better salaries.

(3) Many of the men who have been on leave from their academic positions will need "re-conditioning," in order that they may again take up their research interrupted by the war. It is extremely encouraging to see the number of men who have indicated that they are planning to spend a year or a semester at institutions like the Institute for Advanced Study for this purpose. The Guggenheim Foundation has reserved a number of fellowships for just this purpose and is encouraging outstanding young scholars, who are in need of this opportunity, to apply. Another excellent method would be the creation of more fellowships of the same type as the National Research Council fellowships for post-doctoral research. Anyone who followed the development of mathematics during the period from 1924 to 1935 will realize the tremendous increase in mathematical research among men who had just attained the doctorate and who were given the benefit of these fellowships. This is reflected in the rapid increase in the sizes of our mathematical publications in this period. The *Annals of Mathematics* and the *Transactions of the American Mathematical Society* both increased the sizes of their volumes. The *American Journal*, aided since 1927 by an annual subsidy of $2,500 from the American Mathematical Society, expanded from 300 pages to about 1,000 just before the war. The *Duke Journal*, founded in 1935 by the generous contributions of Duke University, publishes a volume of 800 to 900 pages of high-grade research. The *Quarterly of Applied Mathematics*, founded in 1943, is the latest addition to the list of journals publishing mathematical research.

Unless new personnel is recruited for mathematics by the influx of high-grade graduate students, the whole structure will eventually break down. For the immediate future, this supply must come from those students who had started their graduate work or who had completed undergraduate majors and who were either inducted into the armed forces or entered war work in one of the large physics laboratories, the Aberdeen Proving Ground, the Oak Ridge or Los Alamos Project or others of a similar nature. For these there are available:

(a) The benefits of the GI Bill for men who were in the armed forces.

(b) The pre-doctoral fellowships of the National Research Council. For these fellowships The Rockefeller Foundation has provided a sum of $335,000. They are particularly for men who have been engaged in war work and who are not eligible for the benefits of the GI Bill. There is provision whereby these fellowships may be used to supplement the amount received under the GI Bill. However, in this case, the amount awarded is such as to bring the total amount

available to the student up to the sum of $1,200 plus tuition for an unmarried man and $1,800 plus tuition for a married man. These fellowships are administered by a board set up in the National Research Council under the chairmanship of Professor J. T. Tate, Professor of Physics at the University of Minnesota, and has as members: H. A. Barton, C. W. Bray, D. W. Bronk, L. P. Eisenhart, R. G. Harrison, ex officio, W. A. Noyes, Jr. I was told recently that this committee had already received more than 250 applications. It is evident that this fund will not be able to handle all worthy cases.

(c) Regular university fellowships. A number of universities have not awarded their full quota of fellowships during the war years, because there were not sufficient numbers of candidates available who met their normal standards. These fellowship funds have been allowed to accumulate and consequently at many institutions it will be possible to make more than the usual number of awards in the next few years.

Here there must be a word of warning. Without any doubt, we shall soon be faced with a very serious teacher shortage. Enrollments started to mount very rapidly at the beginning of the present semester. At my own university, we admitted the June high school graduates as freshmen engineers in July and therefore expected a small number in November. Instead, we were required to provide three times as many sections of engineering mathematics as had been expected. There were corresponding increases in liberal arts sections. Another department chairman told me that ten days before college opened, the most liberal estimate as to probable enrollment had been placed at 4,500. The beginning of the semester found that figure had been exceeded by more than 1,000. This state of affairs could be duplicated in many more instances. I predict that the situation with respect to procurement of adequately trained teachers of collegiate mathematics, which is difficult now, will become worse by March and that by September we shall be faced with a shortage more serious than that which faced us in July and August of 1943. The easiest manner of solving our difficulties, at least in part, is to appoint as a full-time instructor the returned service man who is now trying to finish his graduate work. This leaves little time for graduate work and, I contend, is absolutely unfair. We must not overload these men by giving them teaching schedules of 12 to 15 hours but should give them all possible assistance in finishing the work for their degrees. Before these men were inducted, they were not able to do effective graduate work because of heavy teaching loads. Many of these men were deferred because they fulfilled the Selective Service requirement of teaching 15 hours per week. Let us see that they have a chance for proper graduate work when they are released.

One very disturbing factor with respect to these men is the fact that they entered the service toward the end of the war and now have very low point scores. Thus, under normal working of the point system, they will be the last people released, even though they are vitally needed for the restoration of the scientific competence of the nation.

Dr. A. H. Compton, Nobel Prize winner and Chancellor of Washington University, said in a recent radio address:

"It takes 6 years for a capable 18-year-old to train himself for an effective research career in science. Even if we start now, it will be 6 years before we again have a normal supply of young scientists for our universities and industry. Can we afford to wait any longer?"

Admiral Furer has stated: "There is a growing belief that important as it may be to maintain after the war, ground forces and sea forces of a size commensurate with our national responsibilities, it may be even more important to keep the weapons and material which we supply these forces in step with the advances of science. Stocking our arsenals with the weapons of this war is no guarantee that we can win the next war. In fact, it is probably the surest way of losing it. It would be wiser to maintain only arsenals of modest size when we are speaking of ships, or guns or aircraft and to use the money saved thereby to replace the old things with the creations of research laboratory and American genius."

In face of these and many other pronouncements, our promising scientists have not been given preferential demobilization. This state of affairs exists in spite of the fact that neither our allies nor, so far as we know, our enemies have permitted such a condition to develop. On the contrary, they have maintained or increased their programs for the training of scientists and technologists during the war. The War Policy Committee of the Society and Association discussed the desirability of preferential demobilization of prospective graduate students and teachers of mathematics as early as November, 1944. Through the committee, the question was presented to the Office of Scientific Personnel and eventually we placed the problem squarely before the subcommittee, headed by Dr. Henry Allen Moe of the Guggenheim Foundation, which was to study the question of the discovery of scientific talent, placed before Dr. Vannevar Bush by President Roosevelt. I should like to read the recommendations of the Moe committee, made shortly after V-E Day:

"Arrangements should be made now with the Army and the Navy whereby now that it is militarily feasible, these talented students should be ordered to duty in the United States for fully independent, integrated scientific study of a grade available to civilians in peace times. This should be adopted as the considered policy of the armed services and no desire of a commanding officer to retain a potential scientist for his usefulness on the spot should be allowed to interfere with the operation of the policy.

"It is recommended that this plan be carried out, not in terms of a stated number of young scientists, but rather that, now, centers of science and technology in the United States should be combed for information concerning those students who, prior to the war, had given evidence of high talent for science and technology; and that, as soon as militarily possible, these students by name, should be ordered to duty as students. Probably no more than 100,000 of the 10,000,000 men in the Army and Navy would be involved and now, following

V-E-Day, that number could not be militarily significant. Likewise, we recommend that the armed services comb their records for men, who, during the war, have given evidence of high talent for science and technology, and they also be included in this plan."

During the last summer I sent a questionnaire to chairmen of departments of mathematics in colleges and universities on the approved list of the Association of American Universities, asking for the names of promising young mathematicians who should be considered for such a program. More than 450 names were secured and were forwarded to the Office of Scientific Personnel, directed by Dr. M. H. Trytten, who is to prosecute this with the proper authorities.

There is one hopeful sign that no further losses will be incurred in the ranks of those who are now graduate students in science. I have been informed that on November 23 there went to press, for distribution to local boards, a directive providing for the deferment of all graduate students and senior students in physical science and mathematics. This is also to apply to teachers of mathematics. The information which has come to me is that this memorandum will be released on November 27.

In addition to immediate plans, we must look at the long-range picture. In the post-war years there must be a continuous supply of young men capable of entering upon research careers in science. Here there is a very disturbing factor for there has been continually looming on the horizon the spectre of universal military training in peace time. This training period would delay the time when young men could start their scientific careers and many others who would thus be drawn into military service might be lost to higher education entirely. In 1944, the War Policy Committee appointed a subcommittee to study the problem of universal military training. This committee, while making no statement as to whether universal military training should or should not be adopted, made strong pronouncements as to the type of training which should be given to men who could benefit by scientific training and who would be subject to peace time military service. This subcommittee recommended a program with very strong emphasis on science. This report has been approved by the War Policy Committee, the Council of the Society and the Board of Governors of the Association, and will be published in the near future in the *Bulletin*, this MONTHLY, and *The Mathematics Teacher*. We hope that this factor of universal military training will not retard the growth of graduate work in science.

A hopeful sign for the future lies in the interesting recommendations contained in various science bills before Congress at the present time. One of these would create as many as 6000 four-year scholarships to enable young men who show scientific promise to work for the bachelor's degree, in addition to creating a large number of fellowships to enable the recipients to obtain advanced training leading to the doctorate in science and engineering. However, it must be pointed out that all of this is prefaced upon the fact that good training is to be given to these young prospective scientists in our secondary schools. All of us know the lamentable condition which now exists with respect to the teaching of

arithmetic and algebra in our secondary schools. I am afraid that the conditions which were so eloquently described in the famous Nimitz letter still exist and are likely to continue to exist, unless a very vigorous step is taken to see that the level of the teaching of mathematics in the secondary schools is raised. I am not pleading for an increased amount of required mathematics but I do plead that those who wish to take advanced mathematics in secondary schools should be given the opportunity to do so and that that teaching should be done by competent teachers. In this field, this Association and the National Council of Teachers of Mathematics have an opportunity to do pioneer work of a very significant nature.

Likewise, if increasing numbers of persons are to come to college and take science at the collegiate level, we must be certain that the grade of instruction which we offer in these fields is inspiring and sound, so that those with an aptitude for mathematics will be brought to a full realization of its beauty and majesty and will be prepared to enter upon scientific careers. Here again I do not think that we are entirely without fault and I believe that there can be real improvement in the teaching of mathematics at the collegiate level. There should also be a re-studying of our graduate programs to insure the best possible training for our future research men of mathematics. The United States has assumed world leadership in mathematics. That leadership brings responsibilities, both for imparting the results to coming generations and for ever widening the frontiers of mathematical truth. This responsibility faces both the Mathematical Association of America and the American Mathematical Society. We must see that we meet the challenge.

MATHEMATICS AT THE AMERICAN UNIVERSITIES IN EUROPE

THE STUDY CENTER IN FLORENCE

T. A. BANCROFT, Iowa State College

The University Study Center at Florence, Italy was organized by the United States War Department to offer courses of study on the standard university level to army personnel of the Mediterranean Theater of Operations. It was one of three such institutions in Europe, the other two being at Shrivenham, England and Biarritz, France. The Center was situated about 3 miles from downtown Florence in buildings formerly used by a Fascist School of Aeronautics. The buildings were completed in 1938 and are of brick, stone, and concrete in modern functional design. The eight principal buildings were re-named for American colleges and universities as follows: Princeton Hall, Harvard Hall, Stanford Hall, Yale Hall, Citadel Hall, Cornell Hall, Vanderbilt Hall, and Duke Hall.

The staff at the Study Center was composed of both military and civilian personnel. The military personnel was in complete charge of administration and

also did a large share of the actual classroom teaching. The military personnel was made up of both officers and enlisted men. One rule in the school catalog stated "for purposes of disciplinary control in the classroom, the instructor of whatever rank or grade is the representative of the Commanding General and will be accorded the respect and courtesy of an officer of the highest rank present. The instructor is in absolute authority during the period of his control of a classroom and in respect to the work in his course." For reasons of expediency the first man to arrive at the Study Center in each department was made the head of that department. This meant that all departments were headed by military personnel, since the civilian instructors did not arrive until the organization had been completed. All members of the teaching staff, no matter what their previous rank or grade, were given the title of instructor. Members of the Mathematics Department were as follows:

Military Personnel

H. M. Anderson, 1st Lt., F.A., Thomas Anderson, Cpl., F.A., Stephen Dickler, 1st Lt., S.C., Thomas Dyer, Pfc., Inf., Edmund Fontanella, 1st Lt., Inf., Head of the Department, William Lassow, T/Sgt., F.A., Theodore Love, 1st Lt., C.E., K. O. May, 2nd Lt., Inf., E. Mehr, S.K.3.C., U.S.N.R., William Prahl, 1st Sgt., Inf., Mathew C. Resick, Capt., F.A., Edwin Shotland, T/Sgt., A.C., Warman Welliver, Capt., F.A.

Civilian Personnel

T. A. Bancroft, E. E. Blanche, H. C. Christofferson, W. H. Fagerstrom, Chester Jaeger. Pomona College.

Courses offered in Mathematics were as follows: solid geometry, college algebra, trigonometry, analytic geometry, slide rule, differential calculus, integral calculus, statistics, and a reading course for advanced undergraduate students. One year and a half of high school algebra was made the prerequisite for college algebra. Plane geometry and college algebra were made the prerequisites for trigonometry. Trigonometry was made the prerequisite for the slide rule course. The reading course for advanced undergraduate students was handled on an individual basis. Topics studied included: statistics, probability, projective geometry, advanced calculus, and complex variable. It was found that most of the advanced undergraduute students had been studying mathematics on their own initiative, during spare time, at their home military unit.

The Study Center at Florence ran for four sessions, from July 2, 1945 to November 17, 1945. A session lasted for four weeks, with classes meeting one hour each day for five days a week. The instructor was asked to evaluate each student's work at the end of each week by means of tests, problems to be handed in, or a combination of these two. A final examination was given at the end of each session. Certificates of course completion with letter grade earned

were given the students at the end of each session. The school catalog defined the letter grades as follows: "A" representing an achievement surpassing that of 95% of a normal student body, "B" inferior in achievement to "A" but surpassing that of 70% of a normal student body, "C" inferior in achievement to "B" but surpassing that of 30% of a normal student body, "D" inferior in achievement to "C" but surpassing that of 5% of a normal student body, "E" representing the achievement of the lowest 5% of a normal student body. No student was allowed to remain for a second session whose scholastic record showed an "E" in more than one course. A maximum enrollment of 2700 students was attained during the third session.

Standard text books, prepared for the War Department by well-known writers of mathematical texts, were issued to the students free of charge. Since it was impossible to cover the usual college semester's or quarter's work in one session, students were given a chance to continue a course from one session to the next session provided as many as five enrolled for the second session's work. As a result of this, two sessions of work were offered in most of the mathematics courses, and in some courses three sessions of work were offered. By this means and in such courses, an attempt was made to give a student the necessary background for a subsequent course in college mathematics in the American college or university of his choice. The University Study Center, however, issued no transferable semester or quarter hours of credit; it being left up to the returned veteran's college or university to evaluate the certificate of course completion.

A number of G.I.'s came to the Study Center in order to review previous college courses taken one to four years in the past. Most of these students expected to enter some American college or university upon being discharged. In each course in mathematics students were sectioned according to whether they were reviewing the subject or were taking it for the first time. No attempt was made to section students according to ability or past performance.

It was found in the statistical study entitled, "Soldier Attitudes Toward the University Study Center," prepared for the Commandant, University Training Command MTOUSA, by the Research Branch, Information and Education Section, Hq., MTOUSA; that the students, for the most part, were pleased with the work done at the University Study Center at Florence. A number of the members of the Mathematics Department expressed themselves as pleased with the interest and industry of their students. For the most part they took advantage of this chance to further their intellectual and professional training. Also, as one remarked, thinking about mathematical problems left few brain cells with which to worry about the sweating out of the trip home. Everything that was possible was done by the administration and the faculty to make the University Study Center like a typical American college or university. Probably the Study Center's biggest contribution was to get the G.I.'s back into college patterns of behavior, *i.e.* attending classes, studying, thinking for himself, *etc.* At least, some success was attained in these directions.

BIARRITZ AMERICAN UNIVERSITY

R. M. WINGER, University of Washington

Among the distinctive features of the American University located at Biarritz, seaside resort of romance and glamor, may be mentioned: (1) It is a general educational institution of college grade, comprising schools of Liberal Arts, Science, Engineering, Education, Commerce, Agriculture, Fine Arts, and Journalism, operated directly by an arm of the U. S. Government. (2) It is located in a foreign country. (3) The faculty and students with a few exceptions are Americans in uniform, the secretaries are British girls in the governmental service, the waiters in the dining halls and the technical workmen are French civilians, and the janitors are German prisoners of war. (4) The school sprang full fledged into being with an enrollment of 4,000 students. (5) There is no campus as such, there are no college colors, no college songs and no college yells, but we have a band and an orchestra and publish a newspaper five days a week and operate a theater and radio station. In addition the university enjoys a number of unusual cultural advantages including some of the finest French operas, concerts and lectures, as well as unique Basque festivals and sports.

The difficulties of transforming a beach resort of fashionable hotels and villas into residence halls, class rooms and laboratories were tremendous, especially when it is recalled that the town had been under German occupation for several years and that materials and supplies of all kinds had to be brought in from the United States or Germany while the war in the Pacific was at its height. The ingenuity of branch heads and supply sergeants was taxed to the limit. Thus, painted plywood was used for black boards, while bits of carpet tacked to wooden blocks served as erasers. But through the united efforts of the Army and civilian faculty—and to the surprise of many—the "youngest university in the world" opened its doors on schedule, August 20, 1945.

The following men comprised the mathematical staff for the first term: J. I. Tracey (Branch Head), Frank Ayres, F. R. Bamforth, I. A. Barnett, Lt. Col. A. C. Cohen, Lester Hill, Capt. A. O. Lindstrum, R. B. McClenon, E. R. C. Miles, J. H. Neelley, A. L. Nelson, C. G. Phipps, J. C. Polley, B. P. Reinsch Capt. W. T. Scott, C. R. Sherer, Gabor Szego, H. S. Thurston, Lt. Col. A. S. Turner, Major V. R. Walker, R. M. Winger, Lt. Stephen Oselinsky, Lt. H. E. Temmer, Sgt. E. F. Arndt, Cpl. J. C. Gibson, Cpl. S. D. Conte. No fewer than eight of this faculty are department heads in their respective institutions in the United States.

Of the 4,000 students there were some 800 registrations (including duplicates) in mathematics. Courses were scheduled in the usual lower division subjects, but in response to demand, classes were organized in several advanced and graduate courses. The courses, with the number of sections in each were: college algebra (21), trigonometry (12), analytic geometry (8), mathematics of finance (2), spherical trigonometry and astronomy (1), differential calculus (8), integral calculus (4), differential equations (1), higher algebra (1), advanced

analytic geometry (1), higher mathematics for engineers and physicists (2). In addition a few graduate students have engaged in special study under the direction of Dr. Szego. A mathematical seminar, meeting fortnightly, has also been maintained.

After two weeks the students in college algebra were regrouped and a number of sections of intermediate algebra were organized, the better to adapt the work to the varying needs of the students. This proved so satisfactory that for the next term several sections in intermediate algebra and one in beginning algebra and one in plane and solid geometry are scheduled. Otherwise the courses given during the first term will be repeated and opportunity will be offered to qualified advanced students for individual study under supervision.

As a reward for exceptional performance, 80 students or 2% of the whole student body were selected to continue their work in the second term. These were prorated among the several departments according to the number of majors, the quota for science being 13, of which 6 were students of mathematics. For the second term the international aspect of the university has been increased by the admission of 50 British, 50 French and 25 Canadian students—dubbed the "foreign legion" by the Americans. The mathematical faculty remains intact, except for a few of the military members who are being separated, while the enrollment in mathematics exceeds 1,000.

With the close of the second term, this experiment in higher education will pass into history. How shall we appraise the results? First of all it was apparent from the beginning that any apprehension concerning the psychological problems that these battle-weary veterans might present were happily unfounded. While they were drawn largely from combat troops who had seen the toughest fighting from the beach heads of Normandy to the Elbe River, they appeared not too unlike civilian college boys—older to be sure, somewhat more serious, yet with a disconcerting tendency to prefer refresher courses.

It was the policy in mathematics to maintain college standards in the work covered. It is the concensus of opinion of the faculty that the achievement of the students was about on a par with that in civilian colleges. Classes were kept small, ranging in size from 10 to 20. Grades were assigned on the usual five letter basis with about the usual number of failures being recorded.

Aside from the academic results, there is a universal conviction that the intangible values alone have justified this utopian educational venture. It was reassuring to the soldier to find that he is not merely a cog in a military machine, but a human being who is rated, not by a stripe on a uniform, but by his own intellectual capacity and attainments. The association of men from privates to Lt. Colonels in the same classes, some of them taught by privates and sergeants, was a demonstration in practical democracy. To the mathematical faculty, representing a score of colleges and universities, it has proved an unforgettable experience, this teaching of mathematics in storied Biarritz.

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QUATERNIONS AND REFLECTIONS*

H. S. M. COXETER, University of Toronto

1. Introduction. It is just a hundred years since Cayley began to use quaternions for the discussion of rotations. He was followed by Boole, Donkin, Clifford, Buchheim, Klein, Hurwitz, Hathaway, Stringham, and Study. Apparently none of these men thought of considering first the simpler operation of *reflection* and deducing a rotation as the product of two reflections. This procedure will be described in §§3 and 5, and its consequences developed in the later sections.

Every quaternion $a=a_0+a_1i+a_2j+a_3k$ determines a point $\mathbf{P}_a=(a_0, a_1, a_2, a_3)$ in euclidean 4-space, and a hyperplane $a_0x_0+a_1x_1+a_2x_2+a_3x_3=0$. The reflection in that hyperplane is found to be the transformation $x\rightarrow -a\bar{x}a/\mathrm{N}a$. This leads easily to the classical expression

$$x \rightarrow axb \qquad (\mathrm{N}a = \mathrm{N}b = 1)$$

for the general displacement preserving the origin. Cayley obtained this elegant expression by "brute force" as early as 1855. It became somewhat more natural in the hands of Klein and Hurwitz, forty years later. The treatment in §7 will possibly serve to clarify it still further.

We begin with a few algebraic lemmas, mostly due to Hamilton.

2. Elementary properties of quaternions. A *quaternion* is a hyper-complex number $a=a_0+a_1i+a_2j+a_3k$, where a_0, a_1, a_2, a_3 are real numbers ("scalars"), and multiplication is defined by the rules

$$i^2 = j^2 = k^2 = ijk = -1,$$

which imply $jk=i=-kj$, $ki=j=-ik$, $ij=k=-ji$. Thus quaternions form an associative but non-commutative algebra.

It is often convenient to split a quaternion into its "scalar" and "vector" parts:†

$$a = \mathrm{S}a + \mathrm{V}a, \qquad \mathrm{S}a = a_0, \qquad \mathrm{V}a = a_1i + a_2j + a_3k.$$

We define also the *conjugate* quaternion‡

$$\bar{a} = \mathrm{S}a - \mathrm{V}a = a_0 - a_1i - a_2j - a_3k$$

and the *norm*

$$\mathrm{N}a = \bar{a}a = a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

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* This paper is an amplification of an invited address delivered at the annual meeting of the Mathematical Association of America in Chicago on November 24, 1945.

† W. R. Hamilton, Elements of Quaternions, vol. 1, London, 1899, pp. 177, 193. By his special convention, p. 186, Sxy means S(xy), not (Sx)y; similarly for Vxy and Nxy.

‡ Klein's symbol $\bar{a}$ seems preferable to Hamilton's Ka.

In terms of a and $\bar{a}$, we have $Sa=\frac{1}{2}(a+\bar{a})$, $Va=\frac{1}{2}(a-\bar{a})$.

Since $\overline{ii}=-1=i\bar{i}$ and $\overline{ij}=-k=j\bar{i}$, *etc.*, we easily verify that

$$\overline{ab} = \bar{b}\bar{a},$$

whence $Nab=\overline{a}\overline{b}ab=\bar{b}\bar{a}ab=\bar{b}(Na)b=NaNb$. If $Na=1$, we call a a *unit* quaternion. To every non-vanishing quaternion a there corresponds a unit quaternion*

$$Ua = a/\sqrt{Na}.$$

A quaternion p is said to be *pure* if $Sp=0$. Then $\bar{p}=-p$ and $p^2=-Np$. Thus a pure unit quaternion is a square root of -1.

The point $\mathbf{P}_x=(x_1, x_2, x_3)$ in ordinary space may be represented by the pure quaternion $x=x_1i+x_2j+x_3k$. This representation resembles the Argand diagram in the plane, in that the distance $\mathbf{P}_x\mathbf{P}_y$ is given by $\mathbf{P}_x\mathbf{P}_y^2=N(y-x)$. Since

$$xy = -(x_1y_1+x_2y_2+x_3y_3)+\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i+\begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} j+\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k,$$

we see that $-Sxy$ and Vxy are the ordinary "scalar product" and "vector product" of the two vectors $\mathbf{P}_0\mathbf{P}_x$ and $\mathbf{P}_0\mathbf{P}_y$. If x and y are pure *unit* quaternions, then $\mathbf{P}_x$ and $\mathbf{P}_y$ lie on the unit sphere around the origin $\mathbf{P}_0$, and we have $\angle\mathbf{P}_x\mathbf{P}_0\mathbf{P}_y=\theta$, where

$$\cos\theta = -Sxy = -\tfrac{1}{2}(xy+yx).$$

Thus the condition for $\mathbf{P}_x$ and $\mathbf{P}_y$ to lie in perpendicular directions from $\mathbf{P}_0$ is

$$xy + yx = 0.$$

LEMMA 2.1. *For any quaternion a we can find a unit quaternion y such that $ay=y\bar{a}$.*

Proof. Take any pure quaternion p for which $\mathbf{P}_0\mathbf{P}_p$ is perpendicular to $\mathbf{P}_0\mathbf{P}_{Va}$.† Then

$$(Va)p + p(Va) = 0.$$

But since the scalar Sa commutes with p,

$$(Sa)p - p(Sa) = 0.$$

By addition,

$$ap - p\bar{a} = 0.$$

The desired *unit* quaternion is $y=Up$.

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* Hamilton, *op. cit.*, p. 137.

† If an explicit formula is desired, we may write $p=V(Va)q$, where q is any pure quaternion (other than the scalar multiples of Va). In fact, if $p=V\alpha q$, where $\alpha=Va$, then $2p=\alpha q-q\alpha$, and

$$2(\alpha p + p\alpha) = \alpha(\alpha q - q\alpha) + (\alpha q - q\alpha)\alpha = \alpha^2 q - q\alpha^2 = 0,$$

α^2 being scalar. Thus our appeal to geometry could have been avoided.

LEMMA 2.2. *Let a and b be two quaternions having the same norm and the same scalar part.* Then we can find a unit quaternion y such that $ay=yb$.*

Proof. If $b=\bar{a}$, this is covered by Lemma 2.1; so let us assume $b\neq\bar{a}$. Since $Sa=Sb$, we have

$$a+\bar{a}=b+\bar{b}, \qquad a-\bar{b}=b-\bar{a},$$
$$a(a-\bar{b})b=a(b-\bar{a})b,$$
$$a(ab-Nb)=(ab-Na)b.$$

Thus $ac=cb$, where $c=ab-Nb=ab-Na=a(b-\bar{a})$. Since $b\neq\bar{a}$, $c\neq 0$; so we can take $y=Uc$.

LEMMA 2.3. *Any quaternion is expressible as a power of a pure quaternion.*†

Proof. Since $Np^t=(Np)^t$, it will be sufficient to prove this for a *unit* quaternion, a. Since $(Sa)^2-(Va)^2=Na=1$, such a quaternion may be expressed as

$$a=\cos\alpha+p\sin\alpha,$$

where $\cos\alpha=Sa$ and p is a *pure* unit quaternion. Since $Va=p\sin\alpha$, we have $p=UVa$. Since $p^2=-1$, de Moivre's Theorem shows that

$$a^n=\cos n\alpha+p\sin n\alpha$$

for any real number n. In particular, $a^{\pi/2\alpha}=p$, so $a=p^{2\alpha/\pi}$. Thus

$$a=p^t,$$

where $p=UVa$ and $\cos\frac{1}{2}t\pi=Sa$ (so that we may suppose $0\leqq t\leqq 2$).

3. Reflections and rotations in three dimensions. We have seen that $\mathbf{P}_x$ and $\mathbf{P}_y$ lie in perpendicular directions from the origin $\mathbf{P}_0$ if the pure quaternions x and y satisfy the relation

$$xy+yx=0.$$

If $Ny=1$, so that $y^2=-1$, this condition may be expressed as

$$x=yxy.$$

Since $\overline{yxy}=\bar{y}\bar{x}\bar{y}=-yxy$, yxy is pure for *any* position of $\mathbf{P}_x$. Thus the linear transformation $x\rightarrow yxy$ (where $Ny=1$) represents a collineation which leaves invariant every point $\mathbf{P}_x$ in the plane through $\mathbf{P}_0$ perpendicular to $\mathbf{P}_0\mathbf{P}_y$, *i.e.*, the plane

$$y_1x_1+y_2x_2+y_3x_3=0.$$

Moreover, it reverses the vector $\mathbf{P}_0\mathbf{P}_y$:

$$y\rightarrow y^3=-y.$$

* Two such quaternions satisfy the same "rank equation" $x^2-2mx+n=0$, where $m=Sa=Sb$ and $n=Na=Nb$. For a general discussion of the equation $ay=yb$, see Arthur Cayley, On the quaternion equation $qQ-Qq'=0$, Messenger of Mathematics, vol. 14, 1885, pp. 108–112.

† Hamilton, p. 399.

But the only collineation having these properties is the *reflection* in the plane of invariant points. Hence

THEOREM 3.1. *The reflection in the plane* $\sum y_r x_r = 0$ *is represented by the transformation*

$$x \to yxy \qquad (\mathrm{N}y = 1).$$

The product of two such reflections, $x \to yxy$ and $x \to zxz$ (with $\mathrm{N}y = \mathrm{N}z = 1$), is the transformation

$$x \to zyxyz.$$

Accordingly, this is a *rotation*, in the plane $\mathbf{P}_0\mathbf{P}_y\mathbf{P}_z$, through twice the angle $\mathbf{P}_y\mathbf{P}_0\mathbf{P}_z$. In other words, it is a rotation through ϕ about the line $\mathbf{P}_0\mathbf{P}_{\mathrm{V}yz}$, where

$$\cos \tfrac{1}{2}\phi = -\mathrm{S}yz.$$

By Lemma 2.1, any unit quaternion a may be expressed as $-zy$, where $ay = y\bar{a} = z$. (Here z, like y, is a pure unit quaternion; for,

$$z + \bar{z} = ay - y\bar{a} = 0, \quad \text{and} \quad \mathrm{N}z = \mathrm{N}a\mathrm{N}y = 1.)$$

Since $\bar{a} = -yz$, this shows that the transformation

$$x \to ax\bar{a} \qquad (\mathrm{N}a = 1)$$

is a rotation through ϕ about $\mathbf{P}_0\mathbf{P}_{\mathrm{V}a}$, where $\cos \frac{1}{2}\phi = \mathrm{S}a$.

In the notation of the proof of Lemma 2.3, we have $a = \cos\alpha + p \sin\alpha$, where $\mathbf{P}_0\mathbf{P}_p$ is the unit vector along the axis of rotation, and $\alpha = \pm\frac{1}{2}\phi$. The sign is a matter of convention. Taking the plus, we find that the rotation through $\frac{1}{2}\pi$ about the x_3-axis $\mathbf{P}_0\mathbf{P}_k$ transforms $\mathbf{P}_i$ into $\mathbf{P}_j$:

$$i \to ai\bar{a} = \frac{1+k}{\sqrt{2}}\, i\, \frac{1-k}{\sqrt{2}} = \frac{(i+j)(1-k)}{2} = j.$$

(The opposite convention would have given $i \to -j$.) We sum up in

THEOREM 3.2.* *The rotation through* ϕ *about the line with direction cosines* (p_1, p_2, p_3) *is represented by the transformation* $x \to ax\bar{a}$, *where*

$$a = \cos \tfrac{1}{2}\phi + (p_1 i + p_2 j + p_3 k) \sin \tfrac{1}{2}\phi.$$

Since the product of two rotations, $x \to ax\bar{a}$ and $x \to bx\bar{b}$, is another rotation, *viz.*, $x \to bax\overline{ba}$, we can immediately deduce

THEOREM 3.3. *All the rotations about lines through the origin in ordinary space form a group, homomorphic to the group of all unit quaternions.*

Since the rotation through ϕ is indistinguishable from the rotation through $\phi + 2\pi$ about the same axis, there are *two* quaternions, $\pm a$, for each rotation.

* Arthur Cayley, On certain results relating to quaternions, Philosophical Magazine (3), vol. 26, 1845, p. 142. George Boole, Notes on quaternions, *ibid.*, vol. 33, 1848, p. 279. W. F. Donkin, On the geometrical theory of rotation, *ibid.* (4), vol. 1, 1851, p. 189.

Alternatively, we may say that the group of rotations is *isomorphic* to the group of all "homogeneous quaternions," in accordance with the formula

$$x \to axq/Na, \quad \text{or} \quad x \to axa^{-1}.$$

(A homogeneous quaternion is the class of all scalar multiples of an ordinary quaternion.) To make this an isomorphism rather than an anti-isomorphism, we must agree to multiply group elements from right to left.

Combining $x \to axq$ with the *inversion* $x \to -x$, we obtain the transformation

$$x \to -axq \qquad (Na = 1)$$

which may be regarded either as a *rotatory-inversion* of angle ϕ or as a *rotatory-reflection* of angle $\phi+\pi$.

4. The general displacement. Two orthogonal trihedra at the same origin can be brought into coincidence by the successive application of the following reflections: one to interchange the two x_1-axes, another to interchange the two x_2-axes (without disturbing the x_1-axis), and a third (if necessary) to reverse the x_3-axis. Thus the general orthogonal transformation in three dimensions is the product of at most three reflections—an even or odd number according as the transformation preserves or reverses sense. In particular, any *displacement* (or "movement") leaving the origin fixed must be the product of only two reflections *i.e.*, a *rotation*. (This is a famous result, due to Euler.) Thus the group considered in Theorem 3.3 is the group of all such displacements, and is a subgroup of index 2 in the group of all orthogonal transformations. The latter contains also the rotatory-reflections $x \to -axq$, which are products of three reflections.

Similarly in four dimensions, the general orthogonal transformation (*i.e.*, congruent transformation with a fixed origin) is a product of at most four reflections (in hyperplanes). Thus the general displacement (leaving the origin fixed) is the product of two or four reflections. But the product of two reflections is a rotation (about the common plane of the two hyperplanes, through twice the angle between them). Hence a displacement is either a single rotation or the product of two rotations about distinct planes. If any point besides the origin is invariant, the displacement can only be a single rotation; for it operates essentially in the 3-space perpendicular to the line of invariant points. In the general case of a double rotation, where only the origin is fixed, it is well known (though far from obvious) that the axial planes of the two rotations may be chosen to be completely orthogonal. This was first proved by Goursat in 1889. We shall obtain a new proof in §9.

5. Reflections and rotations in four dimensions. A point P_x in euclidean 4-space has four Cartesian coordinates (x_0, x_1, x_2, x_3) which may be interpreted* as the constituents of a quaternion

$$x = x_0 + x_1 i + x_2 j + x_3 k.$$

* A. S. Hathaway, Quaternions as numbers of four-dimensional space, Bulletin of the American Mathematical Society, vol. 4, 1897, pp. 54–57.

The distance $\mathbf{P}_x\mathbf{P}_y$ is given by

$$\mathbf{P}_x\mathbf{P}_y^2 = (y_0 - x_0)^2 + (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 = \mathrm{N}(y - x).$$

If x and y are *unit* quaternions, then $\mathbf{P}_x$ and $\mathbf{P}_y$ lie on the unit hypersphere around the origin $\mathbf{P}_0$, and we have $\angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_y = \theta$, where

$$\cos\theta = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = \mathrm{S}x\bar{y} = \tfrac{1}{2}(x\bar{y} + y\bar{x}).$$

Thus the condition for $\mathbf{P}_x$ and $\mathbf{P}_y$ to lie in perpendicular directions from $\mathbf{P}_0$ is

$$x\bar{y} + y\bar{x} = 0.$$

If $\mathrm{N}y = 1$, this condition may be expressed as

$$x = -yxy.$$

Thus the linear transformation $x \rightarrow -yxy$ (where $\mathrm{N}y = 1$) represents a collineation which leaves invariant every point $\mathbf{P}_x$ in the hyperplane through $\mathbf{P}_0$ perpendicular to $\mathbf{P}_0\mathbf{P}_y$, *i.e.*, the hyperplane

$$y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 = 0.$$

But it reverses the vector $\mathbf{P}_0\mathbf{P}_y$:

$$y \rightarrow -y\bar{y}y = -y.$$

Hence

THEOREM 5.1. *The reflection in the hyperplane* $\sum y_rx_r = 0$ *is represented by the transformation*

$$x \rightarrow -yxy \qquad (\mathrm{N}y = 1).$$

The product of two such reflections, $x \rightarrow -yxy$ and $x \rightarrow -zxz$ (with $\mathrm{N}y = \mathrm{N}z = 1$), is the transformation

$$x \rightarrow z\,\overline{yx\bar{y}}\,z = zyxyz.$$

Accordingly, this is a rotation, *in* the plane $\mathbf{P}_0\mathbf{P}_y\mathbf{P}_z$, through twice the angle $\mathbf{P}_y\mathbf{P}_0\mathbf{P}_z$. In other words, it is a rotation *about* the completely orthogonal plane

$$\sum y_rx_r = \sum z_rx_r = 0$$

through angle ϕ, where $\cos\frac{1}{2}\phi = \mathrm{S}y\bar{z} = \mathrm{S}z\bar{y} = \mathrm{S}\bar{y}z$. This proves

THEOREM 5.2. *The general rotation through angle* ϕ *(about a plane) is*

$$x \rightarrow axb,$$

where $\mathrm{N}a = \mathrm{N}b = 1$ and $\mathrm{S}a = \mathrm{S}b = \cos\frac{1}{2}\phi$.

Conversely, the transformation $x \rightarrow axb$ (where $\mathrm{N}a = \mathrm{N}b = 1$) is a rotation whenever $\mathrm{S}a = \mathrm{S}b$; for then, by Lemma 2.2, we can find unit quaternions y and z such that $ay = yb = z$, enabling us to write

$$a = z\bar{y}, \qquad b = \bar{y}z.$$

6. Clifford translations. The product of the two rotations $x \to axa$ and $x \to axa\bar{a}$ is the so-called *left translation**

$$x \to a^2 x,$$

while the product of $x \to axa$ and $x \to \bar{a}xa$ is the *right translation*

$$x \to xa^2.$$

Clearly, left translations $x \to ax$ form a group, so do right translations $x \to xb$, and any left translation commutes with any right translation. Every left translation has a *unique* expression $x \to ax$ ($Na = 1$); for, the equation $ax = bx$ would imply $a = b$. Similarly every right translation has a unique expression $x \to xb$. Hence

THEOREM 6.1. *The group of left (or right) translations is isomorphic to the group of unit quaternions.*

(In the case of right translations, we can make this a true isomorphism, and not merely an anti-isomorphism, by letting the quaternion b correspond to the translation $x \to x\bar{b}$, so that the product ab corresponds to $x \to xa\bar{b} = xb\bar{a}$.)

Comparing this with Theorem 3.3, we see that the group of left (or right) translations is homomorphic to the group of rotations about a fixed origin in ordinary space, with two Clifford translations for each rotation.

A Clifford translation (*i.e.*, a left or right translation) has the remarkable property of turning every vector through the same angle. For, if $Nx = 1$, so that also $Nax = 1$, the left translation $x \to ax$ transforms $\mathbf{P}_x$ into $\mathbf{P}_{ax}$, and

$$\cos \angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_{ax} = Sax\bar{x} = Sa,$$

which is the same for all vectors $\mathbf{P}_0\mathbf{P}_x$. Similarly, $x \to xb$ transforms $\mathbf{P}_x$ into $\mathbf{P}_{xb}$, and $\cos \angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_{xb} = S\bar{x}xb = Sb$.

Can a left translation be also a right translation? This would make $ax = xb$ for every x. The case $x = 1$ gives $a = b$. Now take x to be the y of Lemma 2.1. Then

$$ya = ay = y\bar{a}, \qquad y(a - \bar{a}) = 0, \qquad Va = 0,$$

and so, since $Na = 1$, $a = \pm 1$. Thus the only left translations that are also right translations† are the *identity* $x \to x$ and the *inversion* $x \to -x$.

Instead of deriving Theorem 5.1 from the condition for $\angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_y$ to be a right angle, we might have observed that $x \to ax$ (where $Na = 1$) must be some kind of congruent transformation (since $Nax = Nx$), and that this transforms the special reflection $x \to -x$ into the general reflection

$$x \to -a\bar{a}x = -a\bar{x}a.$$

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* Felix Klein, Vorlesungen über nicht-Euklidische Geometrie, Berlin, 1928, p. 240.

† William Threlfall and Herbert Seifert, Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes, Mathematische Annalen, vol. 104, 1931, p. 10.

7. The group of displacements. The product of a left translation and a right translation is, of course, $x \to axb$ (where $Na = Nb = 1$). The product of two displacements of this form is another of the same form. In particular, as we saw in Theorem 5.2, a *rotation* is of this form (with the special relation $Sa = Sb$). But the general displacement with a fixed origin is the product of two rotations. Hence

THEOREM 7.1.* *The general displacement preserving the origin is*

$$x \to axb \qquad (Na = Nb = 1).$$

Of course, axb is the same as $(-a)x(-b)$. With this exception, each displacement has a unique expression $x \to axb$. For, the equation $axb = a'xb'$ would imply $a'^{-1}ax = xb'b^{-1}$ for every x, whence $a'^{-1}a = b'b^{-1} = \pm 1$. In other words, every displacement (with a fixed origin) is the product of a left translation and a right translation in just two ways. Thus the direct product of the groups of all left translations and of all right translations is homomorphic to the group of all four-dimensional displacements preserving the origin (with two elements of the direct product for each displacement), and this in turn is homomorphic to the direct square of the group of all three-dimensional displacements preserving the origin (with two displacements for each element of the direct square).

8. The general orthogonal transformation in four dimensions. The general "opposite" or "sense-reversing" transformation leaving the origin invariant is the product of an odd number of reflections. Hence, in four dimensions, it is either a single reflection or a product of three. But in the latter case the three reflecting hyperplanes intersect in a line of invariant points, and every hyperplane perpendicular to this line is invariant; so this scarcely differs from a rotatory-reflection in ordinary space. As such, it has an axis and a special reflecting plane. Its product with the special reflection $x \to -\bar{x}$ is a displacement $x \to axb$; so the rotatory-reflection itself must be $x \to -a\bar{x}b$, or, after changing the sign of a,

$$x \to a\bar{x}b.$$

Since $a\,\overline{a+b}\,b = a+b$, the line of invariant points is P_0P_{a+b}. Since $a\,\overline{a-b}\,b = -(a-b)$, the axis is P_0P_{a-b}.

We sum up our conclusion in

THEOREM 8.1. *Every orthogonal transformation in four dimensions is either*

$$x \to axb \quad \text{or} \quad x \to a\bar{x}b.$$

9. The general displacement expressed as a double rotation. By Theorem 5.2, the general *half-turn* is

$$x \to pxq,$$

* Arthur Cayley, Recherches ultérieures sur les déterminants gauches, Journal für die reine und angewandte Mathematik, vol. 50, 1855, p. 312.

where p and q are pure unit quaternions. This is the half-turn about the plane containing all points $\mathbf{P}_x$ for which $x=pxq$, or

$$px + xq = 0.*$$

Since $p-q$ and $1+pq$ are particular solutions of this equation for x, we may describe the plane as $\mathbf{P}_0\mathbf{P}_{p-q}\mathbf{P}_{1+pq}$. A rotation through $t\pi$ about the same plane is, of course, $x\to p^t x q^t$. Replacing q by $-q$ $(=\bar{q}=q^{-1})$, we deduce that $x\to p^t x q^{-t}$ is a rotation through $t\pi$ about the completely orthogonal plane $\mathbf{P}_0\mathbf{P}_{p+q}\mathbf{P}_{1-pq}$. (The fact that these two planes *are* completely orthogonal is most easily verified by observing that the product of the half-turns $x\to pxq$ and $x\to pxq^{-1}$ is the inversion $x\to p^2x=-x$.)

Thus the product of rotations through $t\pi$ and $u\pi$ about the respective planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$ is $x\to p^{t+u}xq^{t-u}$. Setting $t\pi=\alpha+\beta$, $u\pi=\alpha-\beta$, and observing that

$$\cos\alpha + p\sin\alpha = p^{2\alpha/\pi}$$

(see the proof of Lemma 2.3), we deduce that the product of rotations through angles $\alpha\pm\beta$ about planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$ is

$$x\to(\cos\alpha + p\sin\alpha)x(\cos\beta + q\sin\beta).$$

In other words,

THEOREM 9.1.† *The general displacement $x\to axb$ is the double rotation through angles $\alpha\pm\beta$ about planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$, where*

$$\cos\alpha = \mathrm{S}a, \qquad \cos\beta = \mathrm{S}b, \qquad p = \mathrm{UV}a, \qquad q = \mathrm{UV}b.$$

10. Lines in elliptic space. The above considerations can be translated into terms of *elliptic* geometry by identifying pairs of antipodal points on the hypersphere. Now all scalar multiples of a quaternion x represent the same point $\mathbf{P}_x$, whose coordinates (x_0, x_1, x_2, x_3) are homogeneous. The transformation

$$x\to -y\bar{x}y$$

is the reflection in the polar plane of $\mathbf{P}_y$, and this is the same as the inversion in $\mathbf{P}_y$ itself. We now say that the group of all displacements is precisely the direct product of the groups of left and right displacements; accordingly, it is isomorphic to the direct square of the group of displacements preserving the origin. Instead of Theorem 9.1, we say that the general displacement $x\to axb$ is the product of rotations through angles $\alpha\pm\beta$ about the respective *lines* $\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$, where

(1) $$\cos\alpha = \mathrm{S}a, \qquad \cos\beta = \mathrm{S}b, \qquad p = \mathrm{UV}a, \qquad q = \mathrm{UV}b.$$

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* Irving Stringham, On the geometry of planes in parabolic space of four dimensions, Transactions of the American Mathematical Society, vol. 2, 1901, p. 194.

† Édouard Goursat, Sur les substitutions orthogonales et les divisions régulières de l'espace, Annales Scientifiques de l'École Normale Supérieure (3), vol. 6, 1889, p. 36.

The line $\mathbf{P}_{p-q}\mathbf{P}_{1+pq}$, which is the axis of the half-turn $x \rightarrow pxq$, is conveniently denoted by $\{p, q\}$,* or equally well by $\{-p, -q\}$. Thus *any two pure unit quaternions determine a line* $\{p, q\}$. The absolute polar line is $\{-p, q\}$ or $\{p, -q\}$.

Two lines $\{p, q\}$ and $\{p', q'\}$ have, in general, two common perpendicular lines, which are the transversals of the four lines $\{\pm p, q\}$ and $\{\pm p', q'\}$. These are preserved by either of the half-turns $x \rightarrow pxq$, $x \rightarrow p'xq'$, and so also by their product

$$x \rightarrow p'pxqq'. \tag{2}$$

Thus they are the two axes of this double rotation. Any point on either axis will be reflected first in $\{p, q\}$ and then in $\{p', q'\}$; altogether it will be translated along that axis through twice the distance between the lines. Hence *the two distances between the lines* $\{p, q\}$ *and* $\{p', q'\}$, *measured along their common perpendiculars, are* $\frac{1}{2}|\alpha \pm \beta|$, *where*

$$\cos \alpha = -\mathrm{S}p'p = -\mathrm{S}pp', \qquad \cos \beta = -\mathrm{S}qq'; \tag{3}$$†

and the common perpendiculars themselves are the lines

$$\{\pm \mathrm{UV}pp', \mathrm{UV}qq'\}. \tag{4}$$

It follows from (3) that the condition for $\{p, q\}$ and $\{p', q'\}$ to *intersect* is

$$\mathrm{S}pp' = \mathrm{S}qq', \tag{5}$$

and then the angle between them, being half the angle of the rotation (2), is arc cos $(\pm \mathrm{S}pp')$. Similarly, the condition for $\{p, q\}$ and $\{p', q'\}$ to be *perpendicular* (*i.e.*, for one to intersect the polar of the other) is

$$\mathrm{S}pp' + \mathrm{S}qq' = 0. \tag{6}$$

Thus the condition for them to intersect at right angles is

$$\mathrm{S}pp' = \mathrm{S}qq' = 0. \tag{7}$$

The common perpendicular lines (4) cease to be determinate if $\mathrm{V}pp'$ or $\mathrm{V}qq'$ vanishes, *i.e.*, if either $p' = \pm p$ or $q' = \pm q$. Lines $\{p, q\}$ and $\{p, q'\}$ are said to be *left parallel*. They have an infinity of common perpendiculars

$$\{\mathrm{UV}pp', \mathrm{UV}qq'\}, \tag{8}$$

where p' may range over *all* unit pure quaternions (except $\pm p$). To verify

* This notation was used by A. S. Hathaway, Quaternion space, Transactions of the American Mathematical Society, vol. 3, 1902, p. 53. It is closely associated with the representation of a line in elliptic space by an ordered pair of points on a sphere; see Eduard Study, Beiträge zur nichteuklidische Geometrie, American Journal of Mathematics, vol. 29, 1907, pp. 121–124.

† Here we are using (1) with $a = -p'p$, $b = -qq'$. Plus signs would have given the supplementary distances, which are equally valid; but it seems preferable to use the sign that makes α and β small when p' and q' are nearly equal to p and q.

this we merely have to observe that the line (8), which intersects $\{p, q\}$ at right angles, also intersects $\{p, q'\}$ at right angles. The distance between these left parallel lines, measured along any of the common perpendiculars, is $\frac{1}{2}$ arc cos $(-Sqq')$.

Similarly, *right parallel* lines $\{p, q\}$ and $\{p', q\}$ are distant $\frac{1}{2}$ arc cos $(-Spp')$ along any of an infinity of common perpendiculars (8), only now it is q' that can vary. Thus the common perpendicular lines of right parallels are left parallel, and vice versa.

By the remark at the beginning of §9, the condition for the line $\{p, q\}$ to contain the point $\mathbf{P}_x$ is

$$px + xq = 0. \tag{9}$$

Regarding this as an equation to be solved for q or p, we see that *we can draw through a given point* $\mathbf{P}_x$ *just one left and one right parallel to a given line* $\{p, q\}$, namely

$$\{p, -x^{-1}px\} \quad \text{and} \quad \{-xqx^{-1}, q\}.$$

Thus the set of all left (or right) parallels to a given line is an elliptic congruence: there is just one member of the set through every point of space.

If $\{p, q\}$ contains $\mathbf{P}_y$, its polar line lies in the polar plane of $\mathbf{P}_y$. Replacing q by $-q$, we deduce that the condition for the line $\{p, q\}$ to lie in the plane $\sum y_r x_r = 0$ is

$$py = yq. \tag{10}*$$

Instead of insisting that the p and q of the symbol $\{p, q\}$ shall be *unit* pure quaternions, we could just as well allow them to be any two pure quaternions of *equal* norm. Then $\{p, q\}$ is the same line as $\{\lambda p, \lambda q\}$ for any non-zero scalar λ; *i.e.*, the two pure quaternions are "homogeneous coordinates" for the line. Instead of (3) we must now write

$$\cos\alpha = -Spp'/\sqrt{Npp'}, \qquad \cos\beta = -Sqq'/\sqrt{Nqq'};$$

but instead of (4) we find that the common perpendiculars to $\{p, q\}$ and $\{p', q'\}$ are simply $\{\pm Vpp', Vqq'\}$, or

$$\{Vpp', Vqq'\} \quad \text{and} \quad \{Vp'p, Vqq'\}.$$

Formulas (5), (6), (7), (9), (10) remain valid, but (8) takes the simpler form $\{Vpp', Vqq'\}$.

To express the line $\mathbf{P}_a\mathbf{P}_b$ in the form $\{p, q\}$ we have to find pure quaternions p and q satisfying

$$pa + aq = pb + bq = 0.$$

We may take $p = a\bar{b} - b\bar{a}$ and $q = \bar{a}b - \bar{b}a$, or, halving these, $p = Va\bar{b}$ and $q = V\bar{a}b$. Thus *the line* $\mathbf{P}_a\mathbf{P}_b$ *is* $\{Va\bar{b}, V\bar{a}b\}$.

Similarly, the line of intersection of planes $\sum a_r x_r = 0$ and $\sum b_r x_r = 0$ is $\{Va\bar{b}, Vb\bar{a}\}$.

* Hathaway, Quaternion Space, p. 52.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE EQUATION OF JOUKOWSKI'S AEROFOILS

J. D. MANCILL and BETTY THOMAS, University of Alabama

The conformal transformation

$$W = z + a^2/z,$$

where $z = x + iy$ and $W = u + iv$, and a is any positive real constant, maps suitably chosen circles in the z-plane into curves in the W-plane which resemble the cross section of a modern aerofoil. Cylinders having these images as sections are known as Joukowski's aerofoils. We shall consider in this note the problem of transforming any circle K in the z-plane into a curve in the W-plane by means of this transformation. The equation of this resulting curve L in the W-plane will be shown to be a polynomial of degree four or less. In this way we shall show that the equation of any Joukowski aerofoil section is a polynomial of degree four. There are known methods of constructing the aerofoil point by point for any given circle K as well as approximations to its equation* but so far as the authors know the exact equation is new.

We shall make use of isotropic coordinates in carrying out the transformation. The results show the facility that can be attained in practice by the use of isotropic coordinates in problems of conformal mapping.

Let the circle K in the z-plane be represented by the equation

$$z\bar{z} - \overline{P}z - P\bar{z} + C = 0,$$

where

$$\bar{z} = x - iy, \qquad P = h + ik, \qquad \overline{P} = h - ik, \qquad C = h^2 + k^2 - r^2,$$

and

$$(x - h)^2 + (y - k)^2 = r^2.$$

Therefore, we have the three equations

$$z^2 - zW + a^2 = 0, \tag{1}$$

$$z\bar{z} - \overline{P}z - P\bar{z} + C = 0, \tag{2}$$

$$\bar{z}^2 - \bar{z}\overline{W} + a^2 = 0, \tag{3}$$

from which to eliminate z and $\bar{z}$ and obtain the equation of the curve L in isotropic coordinates.

* See, *e.g.*, Green, Hydro- and Aerodynamics, Pitman, 1937, pp. 62–67.

Substituting the values of $\bar{z}$ from (2) into (3) and simplifying, we obtain

$$rz^2 + sz + t = 0, \tag{4}$$

where

$$r = \overline{P}^2 - \overline{PW} + a^2, \quad s = P\overline{PW} + C\overline{W} - 2\overline{P}C - 2a^2P, \quad t = C^2 + a^2P^2 - PC\overline{W}.$$

The elimination of z between equations (1) and (4) gives the equation

$$\begin{vmatrix} r & s & t & 0 \\ 0 & r & s & t \\ 1 & -W & a^2 & 0 \\ 0 & 1 & -W & a^2 \end{vmatrix} = 0.$$

Expanding this determinant and simplifying the results after substituting the values of P and $\overline{P}$ in terms of h and k, we obtain the equation

$$\begin{aligned} &a_0(W^2\overline{W}^2) + a_1(W^2\overline{W} + W\overline{W}^2) + a_2(W^2\overline{W} - W\overline{W}^2) + a_3(W^2 + \overline{W}^2) \\ &\quad + a_4(W^2 - \overline{W}^2) + a_5(W\overline{W}) + a_6(W + \overline{W}) + a_7(W - \overline{W}) + a_8 = 0, \end{aligned} \tag{5}$$

where a_i, $i=0, 1, \cdots, 8$, are known functions of a, h, k, and C. This is the equation of the curve L in the W-plane.

In order to represent the curve L by means of an equation in the variables u and v, we replace the expressions in the parentheses in equation (5) by their respective values in terms of u and v. This gives the equation we have been seeking,

$$\begin{aligned} &A_0(u^2 + v^2)^2 + A_1(u^3 + uv^2) + A_2(u^2v + v^3) + A_3uv + A_4u^2 \\ &\quad + A_5v^2 + A_6u + A_7v + A_8 = 0, \end{aligned} \tag{L}$$

where

$$\begin{aligned} A_0 &= C(h^2 + k^2), \\ A_1 &= -2(C^2h + a^2h^3 + a^2hk^2 + Ch^3 + Chk^2 + a^2Ch), \\ A_2 &= -2(C^2k - a^2h^2k - a^2k^3 + Ch^2k + Ck^3 - a^2Ck), \\ A_3 &= 8(C^2hk - a^4hk), \\ A_4 &= 5C^2h^2 + C^2k^2 + 2a^2C^2 + 4a^2h^4 + 4a^2h^2k^2 + 5a^4h^2 + a^4k^2 + 6a^2Ch^2 \\ &\quad - 6a^2Ck^2 + a^4C + C^3, \\ A_5 &= 5C^2k^2 + C^2h^2 - 2a^2C^2 - 4a^2h^2k^2 - 4a^2k^4 + a^4h^2 + 5a^4k^2 + 6a^2Ch^2 \\ &\quad - 6a^2Ck^2 + a^4C + C^3, \\ A_6 &= -4(2a^2Ch^3 - 2a^2Chk^2 + 2a^4h^3 - 2a^4hk^2 + a^4Ch + a^6h + C^3h + a^2C^2h), \\ A_7 &= -4(2a^2Ch^2k - 2a^2Ck^3 - 2a^4h^2k + 2a^4k^3 + a^4Ck - a^6k + C^3k - a^2C^2k), \\ A_8 &= a^8 - 16a^4h^2k^2 + 4a^6h^2 - 4a^6k^2 + 4a^2C^2h^2 - 4a^2C^2k^2 + 8a^4Ch^2 \\ &\quad + 8a^4Ck^2 + C^4 - 2a^4C^2. \end{aligned}$$

This equation is of fourth degree except when $C=0$ or when $h=k=0$; that is except when the circle K passes through the origin or when its center is at the origin. When $C=0$, h or k not zero, the equation (L) is of third degree. When $h=k=0$, and $C\neq 0$, it is of second degree and in general represents an ellipse. For the circle $|z|=a$ the ellipse is degenerate, that is the segment of the u-axis from $u=-2a$ to $u=+2a$.

According to Green,* when the circle K passes through $A(z=a)$ and just encloses $B(z=-a)$ the form of the curve L is that of an aerofoil. The shape and dimensions of the aerofoil are dependent upon the position of the circle K. When the circle is fixed the elements h, k, and r are known and therefore the A_i, $i=0, 1, \cdots, 8$, are known functions of these elements. Therefore, the equation of any Joukowski aerofoil section can be determined and it is an equation of the form (L) wih $A_0\neq 0$.

NOTE

MICHAEL WILENSKY, Cincinnati, Ohio

The line of reasoning in my paper, *A Representation of Integers*,† can be simplified and generalized. We had there a formula (3)

$$m = k + k' + 2\sqrt{kk' + r}, \qquad m > 6,$$

where $k'>k>0$, $kk'+r>0$, $m>r$, and in the case $r<0$, $m>|2r|$, and therefore $k<m/4$. It is evident that this formula is in general a *recurrence formula*, because k' has also r as quadratic residue and k is in general the smaller coefficient of k' in the equations

$$q^2 = kk' + r \quad \text{and} \quad q'^2 = k''k' + r.$$

The radicand itself is evidently the q of k'. One can, therefore, consider k', k'' as numbers m of different orders, or as functions of the place "t" occupied by them: $m(t)$, $m(t-1)$, $m(t-2)$, $\cdots$. Thus (3) can be written

$$m(t) = k + m(t-1) + 2q(t-1).$$

Proceeding thus further, we shall finally arrive, since the numbers m are decreasing with diminishing t whereas k and r remain constant, at a number m, $m(\lambda)$, which will not satisfy the required conditions. The smallest m which satisfies these conditions we denote by $m(0)$, but the series has no greatest number; it is an infinite one. We have accordingly

$$\begin{aligned} m(0) &= m(0) \\ m(1) &= k + m(0) + 2q(0) \\ m(2) &= k + m(1) + 2q(1) \\ &\cdots\cdots\cdots\cdots \\ m(t) &= k + m(t-1) + 2q(t-1). \end{aligned} \tag{1}$$

* Loc. cit., p. 66.

† This MONTHLY, vol. 51, 1944, pp. 466–468.

Formula (4) of my previous paper

$$q = k + \sqrt{kk' + r}$$

can be written (see above)

$$q(t) = k + q(t-1). \tag{2}$$

By the way, it follows from equations (1) and (2) that the constant second differences of the terms of the series, $[m(t)-m(t-1)]-[m(t-1)-m(t-2)]$, equal $2k$.

By induction we find from (1)

$$m(t) = kt^2 + 2q(0)t + m(0),$$

and from (2)

$$q(t) = kt + q(0) = m'(t)/2,$$

and hence

$$m(0) = kt^2 - 2q(t)t + m(t).$$

Of course, there can be, and, in case of composite k, must be other initial numbers with the same k and r which constitute different series.

Example. Let $m(0)=9$, $q(0)=4$, and thus $k=1$, $r=7$; $m(1,000)=1,008,009$, $q(1,000)=1,004$, and

$$1,008,009 = 1 + 1,006,002 + 2\sqrt{1,006,002 + 7}.$$

It is, however, not necessary to take $m(0)$ initially; one can take $m(\lambda)$ and compute in the same manner $m(\lambda \pm t)$, where $\lambda - t \geqq 0$.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944-45

Pi Mu Epsilon, Michigan State College

The new officers who took office in April 1944 sent the club members on a scavenger hunt at the first meeting to bring back ten specified objects defined cryptically or in mathematical terms, including a cone and arcsin (1/2). These two were combined as refreshments after they had been produced for the judges. Other programs were as follows:

Magic squares, by Janey Humphreys, Jean Bullard, and Wendel Blanding

Tricks with the slide rule, by Richard Graves and Sterling Berberian

How smart are the professors? with boners illustrated by members of the Mathematics Department and detected by the students

Lives of great mathematicians, a series of five minute talks given by eight new student initiates

Probabilities in dice, poker, and bridge, by Mrs. Mary Jean Thomas, Rosa Sorenson, and Beva Dirksen

Machines for solving algebraic equations, by Professor J. S. Frame

Numbers are peculiar animals, a series of five minute talks by members of the Department

Some mathematical curves, a series of five minute talks by each of the newly elected student initiates

The date of Easter, by Thera Chiaverini, and

Continued fractions, by Mary Mortimer

The calculus method of comparing means, by Jean Bullard, and

The effect of outside work on grades, by Eleanor Treat.

Two Louis C. Plant scholarship awards were presented at the March banquet to outstanding Juniors majoring in mathematics and deemed most worthy on the basis of scholarship, interest in mathematics, and helpfulness to the Department of Mathematics. Jane Walker received a first prize of fifty dollars, and Eunice Anderson a second prize of twenty-five dollars. A quiz program followed the presentation of awards. Picnics were held in the East Lansing State Park in the spring and in the Forestry Cabin in the fall.

Officers for the year 1944–45 were: President, Mary Jean Armstrong (later Mrs. Thomas); Vice-President, Rosa Sorenson; Secretary-Treasurer, Beva Dirksen. Officers for 1945–46 are: President, Rosa Sorenson; Vice-President, Eunice Anderson; Secretary-Treasurer, Jane Walker.

Archimedeans, Winthrop College

The following talks were presented during the past year:

Men who discovered conic sections, by Katherine Lawrence

Conic sections, by Dr. Ruth Stokes

Famous women in mathematics, by Gea Dealia Wolfe and Theora Bernstein

Outstanding women in the field of mathematics, by Emily Boone, Frances Dantzler, and Althea Floyd

The origin of arithmetic, by Hester Carpenter, Ruth Cottingham, and Polly Burgess.

The initiation banquet and party were held at the college shack on October 12, 1945. Twenty-five new members were initiated during the 1945–46 session. Officers for the 1945–1946 session are: President, Althea Floyd; Vice-President, Laura Jean Newell; Secretary, Emily Boone; Treasurer, Frances Dantzler; Social Chairman, Joyce Hutto; Reporter, Frances Crosby; Faculty Sponsors, Mrs. Blanche Badger and Dr. Josephine Mitchell.

Zeno Club, Alfred University

Early in the year the former *Math Club* adopted a constitution which provided for a new name, the *Zeno Club*. Several student and faculty papers were presented as follows at the semi-monthly meetings:

Trick methods of squaring numbers, followed by a discussion of trick methods of multiplying numbers, by Henry Beerman.

Are imaginary numbers real? by Dr. L. L. Lowenstein

The paradoxes of Zeno, by Jewel Karpel

Testing our knowledge, a talk on residues, by Dean A. E. Whitford

Fermat's last theorem, by Virginia Larson

Boolean algebra, a humorous lecture, by Professor W. V. Nevins

Trisection problem, by Betty Lou Fontaine

Nature and mathematics, including a mention of early counting, by Registrar W. A. Titsworth

Radical axis, by Marion Miller

Why I changed from mathematics to education, by Dr. Joseph Seidlin.

For the first meeting prospective club members were the guests of Dr. and Mrs. Lowenstein, and other social functions followed during the year. The club purchased for the library a collection of *University Mathematical Texts*, edited by Alexander Aitken and D. E. Rutherford. Virginia Larson was tapped for membership in Alfred's mathematics honorary, *Pi Delta Mu*.

Officers for the year 1944–45 were: President, Virginia Larson; Vice-President, Henry Beerman; Secretary-Treasurer, Betty Lou Fontaine; Press Agent, Olive Cohen. Officers elected for 1945–46 are: President, Julianne Sanford; Vice-President, Marion Miller; Secretary-Treasurer, Olive Cohen.

Mathematics Club, University of Nevada

Probability and its uses was the subject around which most of the talks to the Mathematics Club of the University of Nevada were given during the fall of 1944 and the spring of 1945. The students of the Mathematics Seminar Class under the guidance of the Faculty Adviser, Dr. E. M. Beesley, were responsible for the organization program.

The following lectures were given:

Pari-mutuel betting, by Charlotte Ferris

Combinatorial analysis, by Katherine O'Leary

Mathematics in army engineering, by Major E. C. Grafton

Some Diophantine problems, by Dr. E. M. Beesley.

At the meeting devoted to entertaining the freshmen mathematics students, a written quiz contest was held, and prizes were given to students obtaining highest scores. Major E. C. Grafton of the Civil Engineering Department and veteran of the Normandy beachhead was the guest speaker. Two highly successful social events were held: the annual Christmas Party at the home of Dean Fredrick Wood, and the Spring Picnic held jointly with the Chemistry Club and Alpha Epsilon Delta. The 1944–45 officers were: President, Mary Watts; Vice-President, Katherine O'Leary; Secretary-Treasurer, Margie McQuerry.

Harvard Mathematical Club, Harvard University

Meetings of the club were frequently attended during 1944–45 by visitors from Boston University, M.I.T., Regis, and Tufts. Papers presented were:

How to skate; or, the Laplace integral, by Professor H. Whitney

The four-color problem, by Arthur Greenwood

Transfinite arithmetic, by Roger Lyndon

The mathematical theory of international politics, by Dr. R. P. Boas, Jr.

Rotation and projection, by John Breakwell of Tufts College

Language and mathematics, by Carl Cohen

Non-euclidean geometry, by Richard Arens, first prize winner

Elementary algebraic geometry, by Dr. I. S. Cohen

The Banach-Tarski paradox; or, it's a small world, by E. J. Akutowicz

Map projections, by Lindley Burton, second prize winner

e^{e^x}; *or, analysis with e's*, by G. T. Williams, subsequently published in this MONTHLY, vol. 52, 1945, pp. 323–327, under the title *Numbers generated by the functions* e^{e^x-1}

Some freak functions, by Professor A. Zygmund of Mount Holyoke College

Trivial summability, by R. C. Buck

Some borderline cases in astronomy and mathematics, illustrated by lantern slides, by Dr. Herbert Jehle of the Physics department

The axiom of choice, by Henry Helson

Dimensional analysis, by R. W. Landauer

Elementary Diophantine approximation, by Professor R. Salem

The probability of occupancy; or, how many people have the same birthday, by Professor R. von Mises.

Recipients of the Robert Fletcher Rogers Prizes for the best student papers were Mr. Arens ($35) and Mr. Burton ($15). At the last meeting the club voted to present to the Department of Mathematics a portrait of the late Professor G. D. Birkhoff. Officers for 1944–45 were: President, Mrs. Ellen Buck; Vice-President, R. F. Arens; Secretary, J. A. Zilber; Treasurer, R. C. Buck; Faculty Adviser, Dr. R. P. Boas, Jr.

Mathematics Club, Wellesley College

The Wellesley College Mathematics Club held two meetings during the year 1944–45. The number of meetings had been reduced to allow students more time for participation in war activities. The first meeting was devoted to student reports on the topic

Mathematical fallacies.

A guest speaker was invited for the second meeting, at which the Club heard an address entitled

The Jewish Passover and the Christian Easter, by Professor Bancroft Brown of Dartmouth College.

The officers of the club were: President, Patricia Boland; Vice-President, Mrs. Joy Levy; Treasurer, Jeanne Garcelon; Junior Executive, Ida Harrison; Secretary, Miriam Paul.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics of Finance. By T. E. Raiford. Boston, Ginn and Co., 1945. Without tables: 3+176 pages. $2.50. With tables: 3+176+305 pages. $3.60.

This text is intended for a first course in the subject and it consists of two parts. Part I covers the subjects of interest and annuities certain, and Part II is a brief discussion of the simpler forms of life insurance.

The technical mathematical background required from the reader of this text does not go beyond the elementary part of a course in College Algebra. However, a successful student of the text must have the maturity to learn and appreciate the meaning of the highly technical and detailed symbolism which is used; for example, the double superscript notation which is used in the annuity symbols.

Part I begins with the development of the elementary formulas involved in the computation of simple interest. The usual subjects of compound interest, annuities, amortization and sinking funds, building and loan associations, valuation of bonds, and depreciation, follow. There is also a chapter on approximations and limits in financial problems. The approach is logical and the method of presentation is clear. In most cases the general formulas are presented first and then the simpler ones are pointed out as special cases. Numerous illustrative examples of typical problems are worked out in the text to demonstrate how the various formulas are used in their solutions. The lists of exercises seem quite adequate. Indeed the content of Part I appears to be well tested class room material.

Among the noteworthy features are an excellent chapter on annuities, the treatment of loans by the direct reduction plan, and the illustrative tables worked out to show such things as amortization schedules, investment schedules for bonds, *etc.* Also the set of Glover's tables which the book contains is a most valuable and useful asset.

In Part II the author does not presuppose even an elementary knowledge of probability. This necessarily makes the work on life insurance very elementary and somewhat unsatisfactory. However, this will not detract much in using the book as a text for a course in mathematics of finance since, as the author states, only a few lessons can be given to the discussion of the simpler forms of life insurance.

The appearance and typography of this text is in keeping with the usual good standard of Ginn and Company's books.

W. W. ELLIOTT

Introduction to Non-Euclidean Geometry. By H. E. Wolfe. New York, The Dryden Press, 1945. 13+247 pages. $4.50.

In writing this book with the avowed purpose of introducing students whose mathematical training has not been extensive to non-Euclidean geometry, the author is breaking new ground. In many respects he has done an excellent job for he possesses an easy and readable style, and has devoted much time and thought to the subject. The table of contents is as follows: I, The Foundations of Euclidean Geometry; II, The Fifth Postulate; III, The Discovery of Non-Euclidean Geometry; IV, Hyperbolic Plane Geometry; V, Hyperbolic Plane Trigonometry; VI, Applications of the Calculus to Hyperbolic Geometry; VII, Elliptic Plane Geometry and Trigonometry; VIII, The Consistency of the Non-Euclidean Geometries.

In the first three chapters the reader will find an excellent account of the historical background which is particularly important here. The gradual development of ideas throughout the centuries is dramatically told, and a student cannot but be impressed by the significance of the story for the whole of mathematics. It might have been desirable to give a somewhat more detailed account of developments since the time of Riemann. For example, by considering the totality of lines and planes through a point in Euclidean space and their intersections with an arbitrary plane, the fundamental significance of the ideas of projective geometry can be explained in an elementary fashion. From this point of view the whole question of parallelism takes on a new and simpler character.

The development of Hyperbolic geometry in chapter IV is wholly synthetic and is based on the assumption that through a given point more than one line can be drawn coplanar with a given line but not meeting it. This reviewer approves of the lack of emphasis on axiomatic foundations in a book of this character. Ideal points are introduced in §39 and ultra-ideal points in §46. It is here, however, that the approach encounters difficulties. Some reference to projective ideas, possibly along the lines suggested above, would enable the student to see the need for such *inaccessible* points.

The introduction of a metric in chapter V and again in chapter VIII is a ticklish process, especially since the author does not assume a knowledge of analytical methods on the part of the reader. (Cf. Klein's beautiful treatment of the subject in his *Nicht-Euklidische Geometrie*, Springer, 1928.)

The question, then, resolves itself into a pedagogical one. To what extent can the ideas of non-Euclidean geometry be successfully transmitted to students whose geometrical background is limited? Undoubtedly, something can be done and Professor Wolfe deserves a vote of thanks for writing an interesting and readable book. Whether one agrees with his choice of the stage at which the attempt should be made or not, it is clear that this book will stimulate interest in modern geometry which far surpasses that of Euclid in its power to satisfy the aesthetic senses and provide food for philosophical thought.

In an Appendix the author gives the propositions of Euclid Book I; a summary of the properties of the circular and hyperbolic functions and their geo-

metrical interpretations; the theory of orthogonal circles, and an account of the inversion transformation,—all of which are pertinent to the subject in hand.

The book is well printed and bound and the diagrams are particularly well drawn. The proof has been carefully read, for there are no noticeable misprints.

G. de B. Robinson

NEW BOOKS RECEIVED

Industrial Algebra and Trigonometry with Geometrical Applications. By J. H. Wolfe, W. F. Mueller, and S. D. Mullikin. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+389 pages. $2.20.

Sequential Analysis of Statistical Data: Applications. Prepared by Statistical Research Group, Columbia University. New York, Columbia University Press, 1945. 315 pages. $6.25.

PROBLEMS AND SOLUTIONS

Edited by Otto Dunkel, Orrin Frink, Jr., and Howard Eves

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 711. *Proposed by H. S. M. Coxeter, University of Toronto*

Suppose that the vertices of a polyhedron represent places that we wish to visit, while the edges represent the only possible routes. Hamilton considered the problem of visiting all the places, without repetition, on a single journey. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, London, 1939, p. 262.) This is easily solved for the pentagonal dodecahedron. Prove that it cannot be done for the rhombic dodecahedron.

E 712. *Proposed by Donald Eves, Paterson, N. J.*

A man has twelve coins, all of which appear exactly alike, but one of which is counterfeit and does not weigh the same as a genuine coin. He has at his disposal a delicate set of balances, but no weights. How can he detect the false coin, and whether light or heavy, in not more than three weighings? (*Cf.* E 651 [1945, 397].)

E 713. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Find a euclidean construction for a non-regular pentagon which has both a circumcircle and an incircle.

E 714. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration of base B, find the sets of cyclic numbers of three different digits such that the numbers of each set are successive terms of a geometric progression.

If $B=3p+1$ show that the three digits of at least two of these sets are consecutive terms of an arithmetic progression. Apply to the case $B=10$.

E 715. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Suppose $ABCD$ is a proper plane convex quadrilateral and P a point exterior to this plane. Consider the four tetrahedra $PABC$, $PABD$, $PACD$, $PBCD$. If PH is the shortest of all the altitudes of these four tetrahedra, show that H must be interior to $ABCD$. (By a proper quadrilateral is meant one with no three vertices linear.)

SOLUTIONS

Mrs. Miniver's Intersecting Circles

E 681 [1945, 395]. *Proposed by W. B. Campbell, Philadelphia*

From *Mrs. Miniver*: "She saw every relationship as a pair of intersecting circles. The more they overlapped, it would seem at first glance, the better the relationship; but this is not so. Beyond a certain point, the law of diminishing returns sets in, and there are not enough private resources left on either side to enrich the life that is shared. Probably perfection is reached when the area of the two outer crescents, added together, is exactly equal to that of the leaf shaped piece in the middle. On paper there must be some neat mathematical formula for arriving at this; in life, none."

Discuss the possibility of a unique solution for circles of given radii.

Solution by W. W. Johnson, John Huntington Polytechnic Institute. Let the areas of the circles of radii a and b be A_1 and A_2 $(a \leqq b)$, the area of the leaf be L, the area of the crescent of the smaller circle be C_1 and the area of the crescent of the larger circle be C_2. Then $L+C_1=A_1$ and $L+C_2=A_2$, whence $2L+(C_1+C_2)=A_1+A_2$. For the perfection Mrs. Miniver envisions $L=C_1+C_2$, whence $3L=A_1+A_2$ and $L=(A_1+A_2)/3$.

Clearly, the limit of L is A_1, whence $A_1 \geqq (A_1+A_2)/3$, or $A_2 \leqq 2A_1$. Thus it is seen that if the capacity of the more capable of the two individuals concerned is more than twice that of the less capable no ideal relationship can be realized and in case it is exactly twice, the less capable will be completely immersed in the other and no private resources will be left to that member of the pair.

In the general case set $r=b/a$; then $1 \leqq r \leqq \sqrt{2}$ for the ideal condition to exist. Let the circles of radii a and b have centers O_1 and O_2 respectively and intersect in points A and B. Designate angle AO_1B by 2ϕ and angle AO_2B by 2θ.

Then L=sector O_1AB+sector O_2AB−quadrilateral AO_2BO_1, so that for ideal conditions when $3L=A_1+A_2$ we must have

$$L = a^2\phi + b^2\theta - b \sin\theta(a \cos\phi + b \cos\theta) = \pi(a^2 + b^2)/3.$$

Eliminating ϕ by means of the relation

$$a \sin\phi = b \sin\theta,$$

we readily find

$$r^2\theta + \arcsin(r \sin\theta) - r \sin\theta(\sqrt{1 - r^2 \sin^2\theta} + r \cos\theta) = \pi(1 + r^2)/3.$$

This last is the equation that must be solved for θ when the ratio r is given. In a given case this transcendental equation could only be solved by trial involving much time and patience. So that instead of there being a neat mathematical formula for arriving at the solution the mathematics is true to the social problem in presenting a situation of transcendent difficulty for an ideal solution.

In the special case of a perfect mating $r=1$ and our equation becomes

$$2\theta - \sin 2\theta = 2\pi/3.$$

Even this equation is transcendental but not of too great difficulty. Solving this equation by the method of successive approximations, we find $2\theta=149°16.3'$ as the angle at the center of either of the equal circles subtended by one arc side of the common symmetrical leaf.

From the above we can deduce that even when two people are exactly mated the attainment of an ideal relationship is not a mundane affair, but a transcendental problem whose solution, however, can be realized with much less effort and patience than in the case of two people unequally yoked together.

Also solved by Murray Barbour, W. B. Clarke, Constance Davis, W. Unterberg, and the proposer.

Editorial Note: After the proposal of E 681 appeared in print it was brought to our attention that the same problem had been proposed by Otto Gruenberger in the February 1945 issue of the *Graham Dial*, publication of Graham Transmissions, Inc., Milwaukee, Wisconsin. The prize-winning solution was given by W. W. Johnson and appeared in the April 1945 issue of the *Dial*. The above solution is, with a few alterations, the same as that which appeared in the *Dial*.

A Variant of the St. Petersburg Paradox

E 683 [1945, 395]. *Proposed by Irving Kaplansky, Columbia University*

A person tosses a coin n times. For every unbroken run of i heads he will receive 2^i dollars. What is his expectation?

Solution by the Proposer. If the final run of heads is of length r ($r=0, \cdots, n$) the player gains $2^r-2^{r-1}=2^{r-1}$ from the last toss. The probability of this eventuality is $1/2^n$ for $r=n$, and $1/2^{r+1}$ for $r<n$ (for in the latter case the preceding toss must be a tail). Hence, if u_n is his expectation

$$u_n = u_{n-1} + \sum_{r=0}^{n-1} \frac{2^{r-1}}{2^{r+1}} + \frac{2^{n-1}}{2^n}$$
$$= u_{n-1} + n/4 + 1/2.$$

From this and $u_1=1$ we find $u_n=(n^2+5n+2)/8$.

Also solved by D. W. Alling, Murray Barbour, F. W. Herlihy, J. B. Kelly, and E. S. Keeping.

An Extension of E 663

E 685 [1945, 395]. *Proposed by Peter Scherk, University of Saskatchewan*

Prove that the equation $x^r-1=p^n$, where p is a prime while r and n are integers greater than 1, has only one solution. (*Cf.* E 663 [1945, 159].)

Solution by Paul Erdös, Stanford University. We have

$$(x-1)(1+x+\cdots+x^{r-1}) = p^n,$$

whence $x=1+p^t$, $t>0$ (see editorial note). Clearly, then,

$$1+x+\cdots+x^{r-1} \equiv r \pmod{p}, \quad \text{or} \quad r \equiv 0 \pmod{p}.$$

Therefore

$$1+x+\cdots+x^{r-1} = (1+x+\cdots+x^{p-1})(1+x^p+\cdots+x^{n-p}).$$

Hence

$$1+x+\cdots+x^{p-1} = p^u, \quad (u>1, \text{ since } x=1+p^t).$$

Now

$$1+x+\cdots+x^{p-1} \equiv p \pmod{p^t}.$$

Thus $t=1$, or $x=1+p$. But then

$$1+x+\cdots+x^{p-1} = (x^p-1)/(x-1) = [(p+1)^p-1]/p = p^u,$$

or

$$(p+1)^p - 1 = p^{u+1}.$$

Clearly $(p+1)^p-1>p^p$ and $(p+1)^p-1<p^{p+1}$ except for $p=2$, which proves that $(p+1)^p-1=p^{u+1}$ is impossible unless $p=2$. We then have the unique solution $3^2-1=2^3$.

Also solved by D. W. Alling, R. C. Buck, and the proposer. Erdös remarks that the above solution is essentially that given by Gérona in *Nouvelles Annales de Math.*, 9, 1870, p. 469, and 10, 1870, p. 204.

Editorial Note: We may show that $t>0$ in a manner similar to that which the proposer used. Suppose $t=0$, then $x=2$, and $p\neq 2$. Hence $p^2\equiv 1 \pmod 4$ and

$$p^{2k}+1 \equiv 2 \not\equiv 0 \pmod 4, \qquad k>0.$$

Hence n cannot be even. Let n be odd, and let p exp $f(2^r)$. Then $f \mid 2^{r-1}$. On the other hand $f \mid 2n$, for

$$p^{2n} - 1 \equiv (p^n + 1)(p^n - 1) \equiv 0 \pmod{2^r}.$$

Therefore, n being odd, we have $f=1$ or $f=2$. From $p^f \equiv 1 \pmod{2^r}$ it follows that

$$p^f = b \cdot 2^r + 1 > 2^r - 1 = p^n,$$

hence $f > n$. Thus we must have $n=1$, a contradiction of the hypothesis of the original problem.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4193. *Proposed by Hüseyin Demir, Columbia University*

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2} \| A_iA_{i+1}$, and $B_iA_{i+3} \| A_iA_{i+4}$, then the lines A_iB_i concur in a point C.

4194. *Proposed by R. Goormaghtigh, Bruges, Belgium*

In each of the triangles formed by three of the vertices of a cyclic quadrilateral, we consider the projection of the orthocenter on the circumdiameter parallel to the Simson line of the fourth vertex of the quadrilateral with respect to the triangle. The four projections form a quadrilateral inversely similar to the one given and are on a circle concentric to the circumcircle of that quadrilateral.

4195. *Proposed by R. Goormaghtigh, Bruges, Belgium*

There are ten ways to divide six points on a circle into two groups of three so as to form pairs of triangles having no common vertex. The midpoints of the segments joining the orthopoles of a given straight line with respect to each pair of triangles are ten collinear points.

4196. *Proposed by V. Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1 be vertices of equilateral triangles constructed exteriorly, or interiorly, on the sides BC, CA, AB of a triangle, and let A_2, B_2, C_2 be the intersections (BC_1, CB_1), (CA_1, AC_1), (AB_1, BA_1). Then, V and W being the first (or second) centers isogonal and isodynamic of ABC, show that $1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2$.

4197. *Proposed by V. Thébault, Tennie, Sarthe, France*

In a tetrahedron $T \equiv ABCD$ the perpendiculars from an arbitrary point P to the planes of the faces meet again the pedal sphere of P in A', B', C', D'. (1) The intersections of corresponding faces of T and $A'B'C'D'$ belong to a hyperboloid. Generalize. (2) If P is on a sphere with center O and if the perpendiculars to the faces meet it in A'', B'', C'', D'', the orthologic center of T and $A''B''C''D''$, other than P, is the focus of the inscribed paraboloid with the axis parallel to OP.

SOLUTIONS

Tangents to a Twisted Cubic

4141 [1944, 593]. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove synthetically that the four lines of 4018 [1943, 125] are (in general) four tangents to a twised cubic.

Solution by James Jenkins, Student, University of Toronto.

LEMMA 1. *Given a triangle and a point O in the same plane, but not on a side, the locus of trilinear poles of the lines through O is a conic through the vertices of the triangle.*

Proof. If p varies in a pencil, we have, in the notation of the Editorial Note to E 481 [1942, 259], $A_p \barwedge B_p$, whence $P_a' \barwedge P_b'$ and $AP' \barwedge BP'$. Thus P', the trilinear pole of p, traces a conic through A and B (and similarly through C).

LEMMA 2. *If a plane π meets the face BCD of a tetrahedron ABCD in a line a, then the tetrahedral pole of π projects from A into the trilinear pole of a with respect to BCD.*

Proof. Let π meet the four faces $\alpha, \beta, \gamma, \delta$ of the tetrahedron in lines a, b, c, d; let $\alpha', \beta', \gamma', \delta'$ denote the planes Aa, Bb, Cc, Dd; and let A' denote the point $(\beta', \gamma', \delta')$, and so on. Since the tetrahedra $ABCD$ and $A'B'C'D'$ have their four corresponding pairs of faces meeting in coplanar lines, the four lines AA', BB', CC', DD' are concurrent in a point P, which is the tetrahedral pole of π. Let a meet the lines CD, DB, BC in L, M, N (on b, c, d). Then the coaxial planes γ', δ', and APB meet the plane α or BCD in concurrent lines, two of which are CM, DN. So the line (α, APB) joins B to the point (CM, DN). Similarly (α, APC) is $C(DN, BL)$, and (α, APD) is $D(BL, CM)$. These lines thus meet in the trilinear pole of LMN (or a) with respect to BCD. But the planes APB, APC, APD are coaxial in the line AP. Hence the tetrahedral pole P projects into the trilinear pole of a.

LEMMA 3. *Given a tetrahedron and a line o not in a face, the locus of tetrahedral poles of the planes through o is a twisted cubic through the vertices of the tetrahedron.*

Proof. Such an axial pencil of planes meets each face in a flat pencil of lines whose center is not on a side of that face. Hence, by Lemmas 1 and 2, the points of the required locus lie on four quadric cones which join the vertices of the

tetrahedron to conics circumscribing the respectively opposite triangles. These cones evidently have common generators in pairs, so their residual intersection is a twisted cubic through the four vertices.

THEOREM. *A line of general position meets the face-planes of a tetrahedron in four points whose trilinear polars, with respect to those faces, are four tangents of a twisted cubic.*

Proof. The dual of Lemma 3 states that the tetrahedral polar planes of all the points of a line not through a vertex osculate a twisted cubic which further osculates the face-planes of the tetrahedron. As we approach one of the points where the line meets a face-plane, the tetrahedral polar plane tends to coincide with that face-plane, and its limiting intersection is the trilinear polar line of the point in question with respect to that face. From the general theory of envelopes it follows at once that this trilinear polar line is a tangent to the twisted cubic. Hence the four such trilinear polars all touch one twisted cubic.

COROLLARY. *The four trilinear polars touch an infinity of twisted cubics.* See A. C. Dixon, *Quarterly Journal of Mathematics*, vol. 23 (1889), p. 351.

Game of the Red Cross

4148 [1945, 102]. *Proposed by B. M. Stewart, Michigan State College*

A variation of the game of Solitaire is this game of the Red Cross, which is played with checkers on a field of squares arranged in six rows and five columns. During the play there can be at most one piece on a square, and at the outset one square is free so that the play, which consists of a series of jumps, may begin. A jump may be made if in three consecutive squares A, B, C of any row or column there are pieces on A and B while C is free. The jump is moving the piece on A over B to C and removing the piece on B from the play. Jumps along the diagonals of the field are forbidden. The object of the play is to remove all the pieces from the field except five, which are to be left in the form of the Red Cross, as in figure 1.

By the known theory for Solitaire (for example, see this MONTHLY, April, 1941, p. 228) a necessary condition is that the initially empty square be either as in figure 2 or 3. Show that in each of these cases a solution is possible.

Figure 1	Figure 2	Figure 3
x x x x x	o o o o o	o o o o o
x x o x x	o o o o o	o o o o o
x o o o x	o o x o o	o o o o o
x x o x x	o o o o o	o o o o o
x x x x x	o o o o o	o o o o o
x x x x x	o o o o o	o o x o o

Solution by E. D. Schell, Arlington, Va. The 30 squares of the board are numbered consecutively along the rows from left to right. For figure 2 where 13 is the only square without a piece, the moves are as follows:

3 − 13, 6 − 8, 16 − 6, 1 − 11, 26 − 16, 16 − 6, 5 − 3, 3 − 1,
1 − 11, 9 − 7, 18 − 8, 7 − 9, 10 − 8, 11 − 13, 20 − 18, 30 − 20,
15 − 25, 29 − 19, 18 − 20, 20 − 30, 27 − 29, 30 − 28, 28 − 18, 22 − 12.

For figure 3 where 28 is the only square without a piece, the moves are

18 − 28, 25 − 23, 20 − 18, 22 − 24, 12 − 22, 29 − 19, 19 − 17, 27 − 29,
21 − 23, 30 − 28, 28 − 18, 11 − 21, 1 − 11, 26 − 16, 16 − 6, 17 − 19,
10 − 20, 20 − 18, 14 − 12, 3 − 13, 6 − 8, 5 − 3, 2 − 4, 4 − 14.

Solution by the Proposer. For figure 2 the moves are

15 − 13, 24 − 14, 17 − 19, 20 − 18, 30 − 20, 27 − 17, 29 − 27, 26 − 28,
17 − 19, 8 − 18, 19 − 17, 11 − 13, 28 − 18, 18 − 8, 3 − 13, 21 − 11,
1 − 3, 5 − 15, 6 − 16, 16 − 18, 4 − 2, 2 − 12, 20 − 10, 10 − 8.

For figure 3 replace the first seven moves for figure 2 by

30 − 28, 19 − 29, 22 − 24, 25 − 23, 28 − 30, 15 − 25, 30 − 20,

and continue with the remaining moves for figure 2.

Tetrahedron and Powers

4149 [1945, 102]. *Proposed by N. A. Court, University of Okiahoma*

The powers of the vertices of a tetrahedron (T) with respect to the spheres determined by the centroid of (T) and the circles of intersection of the respectively opposite faces of (T) with a sphere (L), of arbitrary radius, concentric with the circumsphere of (T), are equal.

Solution by Howard Eves, College of Puget Sound. Let us take the circumcenter of (T) as origin and let $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ be the position vectors of the vertices and $\mathbf{g}$ the position vector of the centroid of (T). Let R be the radius of the circumsphere and S the radius of (L). Then

$$\mathbf{s}\cdot\mathbf{s} = \mathbf{t}\cdot\mathbf{t} = \mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} = R^2,$$

and the equation of (L) is $\mathbf{x}\cdot\mathbf{x} = S^2$. Now the equation of a general sphere coaxal with (L) and having the plane of the vertices $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ as radical plane is

$$\mathbf{x}\cdot\mathbf{x} + k\,|\,\mathbf{x}-\mathbf{t}, \mathbf{x}-\mathbf{u}, \mathbf{x}-\mathbf{v}\,| - S^2 = 0,$$

where k is a scalar parameter. If we insist that this sphere pass through the centroid $\mathbf{g}$ we find

$$k = (S^2 - \mathbf{g}\cdot\mathbf{g})/|\,\mathbf{g}-\mathbf{t}, \mathbf{g}-\mathbf{u}, \mathbf{g}-\mathbf{v}\,|,$$

giving the sphere

$$\mathbf{x}\cdot\mathbf{x} - S^2 + (S^2 - \mathbf{g}\cdot\mathbf{g})\,|\,\mathbf{x}-\mathbf{t}, \mathbf{x}-\mathbf{u}, \mathbf{x}-\mathbf{v}\,|\,/\,|\,\mathbf{g}-\mathbf{t}, \mathbf{g}-\mathbf{u}, \mathbf{g}-\mathbf{v}\,| = 0.$$

The power of vertex $\mathbf{s}$ with respect to this sphere is then

$$R^2 - S^2 + (S^2 - \mathbf{g}\cdot\mathbf{g})\,|\,\mathbf{s}-\mathbf{t},\mathbf{s}-\mathbf{u},\mathbf{s}-\mathbf{v}\,|\,/\,|\,\mathbf{g}-\mathbf{t},\mathbf{g}-\mathbf{u},\mathbf{g}-\mathbf{v}\,| = R^2 + 3S^2 - 4\mathbf{g}\cdot\mathbf{g},$$

since

$$|\,\mathbf{s}-\mathbf{t},\mathbf{s}-\mathbf{u},\mathbf{s}-\mathbf{v}\,|\,/\,|\,\mathbf{g}-\mathbf{t},\mathbf{g}-\mathbf{u},\mathbf{g}-\mathbf{v}\,| = \text{volume } (\mathbf{stuv})/\text{volume } (\mathbf{gtuv}) = 4.$$

In this manner we see that the four required powers are all equal. The analogous theorem for the plane is also true.

Solved also by the proposer.

Editorial Note. The proposer made the following remark:

An indirect proof of this proposition is contained in my paper, *A porism on eleven spheres*, art. 3b, Mathematical Student, vol. 10, no. 3, Sept. 1942. The proposition is worthy of a direct proof; perhaps one of your contributors will supply one.

The theorem of the problem is true for a simplex σ in euclidean space of n dimensions, the vertices being denoted by B_i, $1 \leqq i \leqq n+1$. Select arbitrarily one vertex as origin O of vectors, then the vectors of the other n vertices A_i are denoted by $\mathbf{a}_i$, $1 \leqq i \leqq n$, and this system of n vectors is linearly independent. There exists an associated system of vectors $\mathbf{a}_i'$ such that $\mathbf{a}_i'\cdot\mathbf{a}_j = 0$, $j \neq i$, $\mathbf{a}_i'\cdot\mathbf{a}_i = 1$; see 3752 [1937, 404]. Denote by $\mathbf{g}$, $\mathbf{c}$ the vectors of the centroid G and the circumcenter C of σ. The equation of the face π_i opposite A_i, of π_0 opposite O, and of the circumsphere (C) are

$$(1) \qquad \pi_i\colon\ \mathbf{a}_i'\cdot\mathbf{x} = 0; \qquad \pi_0\colon\ \mathbf{x}\cdot\sum_{i=1}^{n}\mathbf{a}_i' = 1; \qquad (C)\colon\ \mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} = 0;$$

where $\mathbf{c}^2 = R^2$, the square of the radius of (C). The sphere (L) with center C and radius S has the equation $(\mathbf{x}-\mathbf{c})^2 - S^2 = 0$. The system of spheres

$$(2) \qquad (\mathbf{x} - \mathbf{c})^2 - S^2 + k\mathbf{a}_i'\cdot\mathbf{x} = 0,$$

has the radical plane π_i where k is an arbitrary constant, we determine the one passing through G; and we find that $k = (n+1)\{S^2 - (\mathbf{g}-\mathbf{c})^2\}$. Thus the equation of the sphere through G and the intersection of (L) with face π_i is

$$(3) \qquad (\mathbf{x} - \mathbf{c})^2 - S^2 + (n+1)\{S^2 - (\mathbf{g}-\mathbf{c})^2\}\mathbf{a}_i'\cdot\mathbf{x} = 0,$$

and the power of A_i with respect to this sphere is

$$(4) \qquad P^{(i)}(A_i) = R^2 + nS^2 - (n+1)(CG)^2, \qquad 1 \leqq i \leqq n,$$

after setting $\mathbf{a}_i^2 - 2\mathbf{c}\cdot\mathbf{a}_i = 0$, and $(\mathbf{g}-\mathbf{c})^2 = (CG)^2$. In a somewhat similar manner using the equation for π_0 in (1) we obtain for $P^{(0)}(O)$ the right member of (4); but there is no need for this task, since if we take for the origin any other vertex of σ, we must get again the right member of (4) for the other n vertices. This completes the proof.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Assistant Professor Foster Brooks has returned to Kent State University, Ohio, and has been promoted to an associate professorship.

Assistant Professor G. F. Cramer of Tulane University has been promoted to an associate professorship.

Assistant Professor Douglas Derry of the University of Saskatchewan has been promoted to an associate professorship.

C. B. Helms has been appointed to an assistant professorship at Pennsylvania Military College, Chester, Pennsylvania.

Dr. S. B. Littauer has been appointed chairman of the mathematics department of Newark College of Engineering, Newark, New Jersey.

Assistant Professor A. K. Mitchell of Trinity College, Hartford, has accepted a position in the engineering department of the Pratt and Whitney Aircraft division of the United Aircraft Corporation.

Dr. A. L. Putnam of Yale University has been appointed to an assistant professorship at the College of the University of Chicago.

Dr. Moses Richardson of Brooklyn College has been promoted to an assistant professorship.

C. F. Stephens has returned to Prairie View University, Prairie View, Texas, and has been promoted to a professorship.

Dr. S. L. Thorndike has been appointed to an associate professorship at Winthrop College, Rock Hill, South Carolina.

Professor Oscar Zariski of Johns Hopkins University has been appointed Research Professor of Mathematics at the University of Illinois.

The following appointments to instructorships have been announced:
Brown University: Dr. W. M. Chen
Drake University, Des Moines, Iowa: Dr. A. A. Grau
Johns Hopkins University: Dr. Norman Davids
Ohio State University: Dr. Joshua Barlaz
Oregon State College: Dr. Emund Pinney
University of Michigan: Dr. R. D. Schafer
University of Saskatchewan: W. Buscombe, Dr. W. Fraser, Alma Hiebert, Dr. W. Kozakiewicz, J. S. Vigder
University of Wisconsin: Dr. B. H. Colvin

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE A.A.A.S. COOPERATIVE COMMITTEE ON SCIENCE TEACHING

RALEIGH SCHORLING, University of Michigan

The Cooperative Committee on the Teaching of Science and Mathematics has been reconstituted and is now a standing committee of the A.A.A.S. and operates under the title *The A.A.A.S. Cooperative Committee on Science Teaching.* The new committee met on Saturday and Sunday, October 20–21. The members of the committee, with the organization which each man represents, are as follows: K. Lark-Horovitz, Purdue University, *chairman*, American Association of Physics Teachers; G. W. Warner, Woodrow Wilson Junior College, American Association of Physics Teachers; O. J. Lee, Northwestern University, American Astronomical Society; B. S. Hopkins, University of Illinois, American Chemical Society; L. W. Taylor, Oberlin College, American Institute of Physics; G. W. Blaydes, Ohio State University, Botanical Society of America; L. L. Quill, Michigan State College, Division of Chemical Education, American Chemical Society; E. C. Stakman, University of Minnesota, Executive Committee of the A.A.A.S.; G. A. Thiel, University of Minnesota, Geological Society of America; Raleigh Schorling, University of Michigan, Mathematical Association of America; E. H. C. Hildebrandt, Northwestern University, National Council of Teachers of Mathematics. It is probable that two or three persons, to be selected by several associations not now represented, will presented be added.

The Committee will, in the near future, publish in *School Science and Mathematics* the fourth of a series of reports dealing with the broad problems of science teaching which require collaboration of representatives of the various sciences, entitled "The Preparation of High School Science and Mathematics Teachers." There are five recommendations as follows:

1. *A policy of certification in closely related subjects within the broad area of the sciences and mathematics should be established and put into practice.*

Specifically, any combination of three of the following four subjects is recommended: biological science (including both botany and zoology), chemistry, mathematics, and physics. A combination of general science with any two or three subjects is also recommended.

More and more the colleges have been adopting standards of concentration in a major department which make it difficult and often impossible for an undergraduate to secure a good preparation for teaching in as many as three science fields. While this may be appr priate for the future research scientist, it is not

good for the future high school teacher. The colleges should provide opportunity for a "concentration" which shall lie not in one department but shall spread (not too thinly) over at least three subjects among the sciences and mathematics. The program for the liberal arts degree should allow for such a program of study.

The Committee proposes this policy as a minimum for beginning teachers. As the teacher gains experience and perhaps looks toward a position in a larger school, he should be required to take further work in the subjects in which he plans to specialize.

2. *Approximately one-half of the prospective teacher's four year college program should be devoted to courses in the sciences.*

The Committee recommends that approximately one-half of the four-year college program should be given to courses in science and mathematics. Sixty semester hours, divided among three subjects, will allow for a 24-hour major in one science subject and 18 hours in each of the two others.*

Such a program allows for a reasonable degree of specialization and insures enough breadth to prepare the student reasonably well for the teaching assignments he may expect during his first years of teaching. This program leaves ample room for the usual "general education" requirements of the college and for the required professional courses in education. For the student who plans carefully and is given advice early enough, it even leaves room for 18 or 20 semester hours of a subject such as history or foreign language or music which a student may wish to take as an extra subject to be added to his teaching repertoire.

The Committee is convinced that progress will be made in the preparation of science teachers only if the science departments consider that this preparation is one of their major functions and treat it as such. There should be a section in the college catalog describing the program for the preparation of science teachers. This program should be worked out cooperatively by the science departments and the department of education. Such collaboration will result in a better solution of the problems of student teaching and the courses in methods of teaching, which the Committee recognizes as an essential part of the preparation of a teacher.

The prospective science teacher should make a study of the social consequences and historical backgrounds of science. This may be done through a thorough-going course in this field, offered for junior and seniors, through reading and honors work, and through assisting the instructor of college courses of the "cultural" type. Also the prospective teacher could gain much from observing experimental teaching at the secondary level which aims to correlate and integrate the sciences.

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* When biology is one of the three subjects, the Committee recommends a minimum of 24 hours, rather than 18 hours, because preparation in botany, zoology, and physiology is necessary.

3. *Certificates to teach general science at the 7th, 8th, or 9th grade level should be granted on the basis of not less than 42 semester hours of college courses in the subjects covered in general science.*

Because it provides activities and learning relating science to everyday living in our modern technological world, because it cuts across the boundaries of specialized subject-matter lines, and because it provides an opportunity for all, or nearly all, pupils to explore science phenomena and applications, the Committee recognizes general science as extremely important. It is therefore particularly desirable that a teacher be well equipped to teach such a subject.

The Committee proposes that certificates for teaching general science be granted only to candidates who have college courses in all of the basic sciences. Exception might be made only in the case of people who show proficiency in the broad field of general science through some type of examination such as is suggested later in this report.

The Committee's best guess is that approximately 42 semester hours is an appropriate minimum requirement, with a minimum of 15 hours in biological science, 15 hours in physical science (including both physics and chemistry), and 12 hours in the earth sciences (geology, physiography, meteorology) and astronomy.

The earth sciences should receive more attention in the general science courses. In recent years they have been largely displaced as separate subjects in high-school curricula. The Committee believes that if science teachers had some formal training in that area, there would be a re-introduction of some subject-matter dealing with man's natural environment. Never before has the average person had so wide an opportunity of coming into contact with the diverse features of the earth's surface as is afforded him today through our modern means of transportation, and never before has intelligent citizenship been so much dependent upon understanding of the earth and its natural resources.

4. *Colleges and certification authorities should work toward a five-year program for the preparation of high-school teachers.*

All of the goals of science education would be better achieved if the prospective teacher had a five-year period of preparation. There would then be more time for obtaining the broad liberal education which promotes growth of the student as a person and makes him a better teacher. There would also be time for both the broad training in science which the Committee recommends and the specialization in one subject which is desirable for the teacher in the larger schools.

The only obstacle to the rapid spread of the five-year program is economic. Most of the large cities and several of the wealthier states now require it. Probably the states and areas with less wealth will not be able to pay salaries which justify five years of preparation until legislation is passed to furnish federal aid toward the raising of teachers' salaries.

5. *Curriculum improvements in the small high school should go hand in hand with improvement in teacher preparation.*

If the teachers of science and mathematics were more broadly prepared, they would still have difficulty in the small high school because the number of students is too few to permit the offering of a variety of courses. No school with less than two hundred students, for example, could offer one course in algebra for pupils going to college and a separate course in general mathematics for those not going to college. Yet variety is desirable—at any rate enough variety to provide a sequence of courses for those planning to specialize in the service area and a different sequence for those who will not specialize and in most cases will not go to college. When teachers are broadly enough prepared to teach the variety of courses that are needed, ways must be found in the small school to offer these courses.

One way to solve this problem is to teach two courses simultaneously in the manner employed by every teacher of a one-room rural school. Thus a teacher might, in a single period for ninth and tenth grade pupils, teach algebra to one small group and general mathematics to the others. A teacher of physical science might teach a general course for his entire class and add a couple of periods a week for work on problems and for laboratory work in physics with two or three students who plan to specialize in science in college.

A second suggestion is that the college preparatory courses in mathematics and science be offered in a cycle. Trigonometry and advanced algebra might be offered in alternate years, and similarly with chemistry and physics.

A third suggestion is that the local board of education pay students' tuition for correspondence courses or for Saturday courses taught in larger schools in neighboring cities. Correspondence courses are offered by the extension departments of many state universities, as well as by private schools. Local teachers could easily supervise the work of an occasional student in a correspondence course. The high school in the county seat might offer Saturday courses for pupils from the smaller schools nearby.

The Committee considered several very interesting proposals. Perhaps most significant is a project relating to audio-visual aids. A number of large industrial corporations, on their own initiative, suggested that they were willing to finance a series of movies (both sound and silent) and film strips to aid in the teaching of science and mathematics, and, furthermore, that they were willing to make these available to schools and teachers at low cost. It was further proposed that the Committee contact resourceful teachers and encourage them to write the script for movies and film strips, and that the Committee criticize the scripts, aid the corporations in securing audience responses to be used in the further revision of the films, and, finally, to sponsor the films. The Committee appointed a sub-committee to consider the various proposals.

THE REPORT OF THE HARVARD COMMITTEE

The report of the Harvard Committee on "The Objectives of a General Education in a Free Society," published by the Harvard University Press under the title *General Education in a Free Society*, is a document which every educator should read. Different from many recent studies on general education in the sense that it "presents a view of the total American educational scene," the Harvard Report was developed only after three years of intensive effort on the part of a committee of twelve members. Conferences were held with numerous other persons of wide experience and high standing in the fields of education and business, and a sum of approximately $60,000 was spent upon the undertaking. As President Conant has warned, any judgment of the Report based on incomplete or fragmentary reading is almost certain to be unfair; yet some sections are of sufficient interest to mathematicians that they are reviewed here.

It appears to be the opinion of many critics that the Report's treatment of the problems of secondary education is especially realistic and constructive. Attention is called to the fact that in 1870 three-fourths of those who attended high school went on to college. So the high school's function was clear; it was simply to prepare for college. Now, three-fourths of the high school students do not expect to go to college. This condition raises the fundamental question, "Can the interests of the three-fourths who go on to active life be reconciled with the equally just interests of the one-fourth who go on to further education?" To complicate the problem, "in many states nearly the whole population of high-school age is now in high school, and the same may presently be true of most states. Thus within a generation the problem of how best to meet this immense range of talent and need has grown up, like the fabled beanstalk, to overshadow virtually every other educational problem. It is in truth at the heart of any attempt to achieve education for democracy." Mathematicians are challenged by the assertion that abstract concepts like demonstrative proof in geometry, generalized number in algebra, or hypothesis in science ask for an effort of mind of which many adolescents now in high school seem incapable; a similar statement is made of such general concepts as justice and even democracy. As a result of this condition pictured by the authors of the Report, a plea is made for "even a greater diversity than exists at present in the still largely bookish curiculum" and at the same time for a central course of general education which will give unity to the program of study.

It is proposed that about half, *i.e.*, eight units, of the secondary curriculum of each student be organized as the core of the program of general education. The common and desirable division within the eight units would be three in English, three in science and mathematics, and two in the social studies. The Report is emphatic, however, in insisting that such a program of general education is only a minimum for all students. For those students who will be ending their formal education with high school, another course in each of the three areas seems essential, and for those who are going to college, a deeper knowledge of one or more of the areas is also imperative.

The Report calls attention to the fact that the need of elementary mathematics involves a constantly growing section of the general population. "Beyond this, however, mathematics has an important intrinsic role in general education. It helps build some of the skills and comprehensions that make the effective individual. . . . Mathematics may be defined as the science of abstract form. It is concerned with the universal pattern within the concrete situation." Consequently, mathematics has lost none of its traditional relevance in a general education, though it now must include considerations of its enormous utility in modern life.

It is recommended in the Report that by the end of the seventh or the middle of the eighth grade every student should have acquired a reasonable facility in the language of arithmetic and should possess some competence in the solution of arithmetical problems. Also, by this time, every student should have learned by induction or by intuitive reasoning some of the common facts of geometry. The next stage in the mathematics program, "and the last for those students who are least apt in the subject," should develop some appreciation of and skill in the use of simple algebra, graphs, and trigonometry. "Probably little more than half the pupils enrolled in the ninth grade can derive genuine profit from substantial instruction in algebra or can be expected to master demonstrative geometry. Those who have the requisite abilities should certainly receive such instruction." As a general principle, the Report insists that when a student "has reached his limit of tolerance in handling abstractions, his *general* education in mathematics must also come to an end." For those who show themselves to be inept in mathematics, any further work in the subject might be in the form of shop mathematics, business arithmetic, mathematics of the farm, and so on.

Students who have the capacity for and interest in science and mathematics are encouraged by the Report to continue their work in mathematics by studying a substantial course in algebra and a comparatively rigorous course in demonstrative geometry. Beyond demonstrative geometry, it is recognized that such students would normally study advanced algebra, solid geometry, and trigonometry, but it is urged that consideration be given to the creation of a senior course which "might include—besides elementary trigonometry and some solid geometry—analytic geometry and an elementary approach to the principles of the calculus." It is strongly recommended that provision be made for competent students going into science or mathematics to receive adequate preparation in high school so that the first mathematics studied in college will be the calculus.

In the case of students going to college who have decided that their interest in science and mathematics will not extend beyond a general education in these areas, only two years of high school mathematics is prescribed. However, further mathematics is advised so that the student will have full freedom of choice if he later pursues scientific or technological training. In particular, it is recommended that a new senior high-school course might be created for such students which would include an introductory survey of elementary trigonom-

etry, statistics, precision of measurement, and the use of graphs.

The Report's treatment of the college program is specialized to a consideration of the particular problems of Harvard College. It is concluded that the present Harvard system affords rich opportunities for specialization but general education has been neglected. Consequently, it is proposed that of the sixteen courses required for the bachelor's degree, students should be required to take six courses in general education. Moreover, in any individual program no one of these general courses may be counted for both concentration and general education. Of the six courses, it is specified that at least one shall be in the humanities, one in the social sciences, and one in the sciences. The nature of the courses is discussed in some detail; in general, they are not to be survey courses of the traditional type. In science, for example, it is expected that the general courses should represent reasonably broad syntheses; they should be taught so as "to convey some integrative viewpoint, scientific method, or the development of scientific concepts, or the scientific world-view."

Most of the students now entering Harvard College have studied at least one year of school mathematics beyond the minimal two-year program previously discussed. Consequently, *from the point of view of general education*, the authors of the Report state that "one might be permitted the optimistic view that our students have already completed a minimal program in mathematics before entering college." Moreover, the position is taken that for the student whose work in college makes no specific mathematical demands there is little point, whatever his level of performance, in prescribing remedial mathematics. Since the Report is concerned essentially with the problem of general education, there is no discussion of college courses in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-NINTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-ninth annual meeting of the Mathematical Association of America was held at Chicago, Illinois, on Saturday and Sunday, November 24 and 25, 1945, in conjunction with meetings of the American Mathematical Society. About three hundred and nineteen persons attended the meetings, including the following two hundred and five members of the Association:

L. K. Adkins, State Teachers College, La-Crosse, Wisconsin
A. A. Albert, University of Chicago
G. E. Albert, Ohio State University
N. R. Amundson, University of Minnesota
K. J. Arnold, University of Wisconsin
Max Astrachan, Antioch College
W. L. Ayres, Purdue University

R. W. Babcock, Kansas State College
Ruth M. Ballard, Illinois Institute of Technology
R. H. Bardell, University of Wisconsin in Milwaukee
R. W. Barnard, University of Chicago
C. F. Barr, University of Wyoming
Walter Bartky, University of Chicago
W. D. Baten, Michigan State College
Grace E. Bates, University of Illinois
S. Louise Beasley, Drury College
S. F. Bibb, Illinois Institute of Technology
G. A. Bliss, University of Chicago
L. M. Blumenthal, University of Missouri
C. R. Blyth, University of Toronto
A. O. Boatman, Carthage College
H. F. Bohnenblust, Indiana University
W. M. Borgman, Jr., Wayne University
C. C. Bramble, U. S. Naval Academy
Richard Brauer, University of Toronto
R. H. Bruck, University of Wisconsin
E. L. Buell, Northwestern University
R. S. Burington, Case School of Applied Science
H. E. Burns, Indiana University Extension
I. W. Burr, Purdue University
L. E. Bush, College of St. Thomas

S. S. Cairns, Queens College
W. D. Cairns, Oberlin College
R. H. Cameron, University of Minnesota
H. H. Campaigne, University of Minnesota
C. S. Carlson, St. Olaf College
W. B. Carver, Cornell University
W. F. Cheney, Jr., University of Connecticut
P. L. Chessin, University of Wisconsin
E. W. Chittenden, University of Iowa
Laura E. Christman, Senn High School, Chicago
R. V. Churchill, University of Michigan
Paul Civin, University of Buffalo
E. H. Clarke, Hiram College
C. J. Coe, University of Michigan
Nancy Cole, Connecticut College
J. J. Corliss, DePaul University
Richard Courant, New York University
N. A. Court, University of Oklahoma
H. S. M. Coxeter, University of Toronto
A. E. Currier, U. S. Naval Academy

Flora Dinkines, University of South Carolina
J. E. Dotterer, Indiana Central College
W. L. Duren, Jr., Tulane University
W. H. Durfee, Northwestern University

D. H. Erkiletian, Jr., University of Missouri, School of Mines and Metallurgy
H. J. Ettlinger, University of Texas
H. P. Evans, University of Wisconsin
H. S. Everett, University of Chicago

Edna M. Feltges, Wilson Junior College
L. R. Ford, Illinois Institute of Technology
Tomlinson Fort, University of Georgia
J. S. Frame, Michigan State College

M. P. Gaffney, Jr., University of Chicago
B. H. Gere, U. S. Naval Academy
J. S. Georges, Wright Junior College
J. W. Givens, Northwestern University
A. M. Gleason, U. S. Navy
L. M. Graves, University of Chicago
Edison Greer, University of Kansas

V. H. Haag, Hershey Junior College
D. W. Hall, University of Maryland
P. R. Halmos, Syracuse University
O. H. Hamilton, Oklahoma A. and M. College
M. L. Hartung, University of Chicago
E. D. Hellinger, Northwestern University
R. G. Helsel, Ohio State University
Fritz Herzog, Michigan State College
M. R. Hestenes, University of Chicago
E. H. C. Hildebrandt, Northwestern University
T. H. Hildebrandt, University of Michigan
J. D. Hill, Michigan State College
D. L. Holl, Iowa State College
Sister Mary Charlotte Holland, St. Xavier College
G. B. Huff, University of Texas
R. C. Huffer, Beloit College
H. K. Hughes, Purdue University
Ralph Hull, University of Nebraska
J. A. Hyden, Vanderbilt University

M. H. Ingraham, University of Wisconsin

R. L. Jeffery, Queen's University
B. W. Jones, Cornell University

Irving Kaplansky, University of Chicago
Dora E. Kearney, Iowa State Teachers College

M. W. KELLER, Purdue University
A. J. KEMPNER, University of Colorado
E. C. KIEFER, James Millikin University
J. R. KLINE, University of Pennsylvania
W. C. KRATHWOHL, Illinois Institute of Technology
JOHN KRONSBEIN, Evansville, Indiana
W. H. KURZIN, Chicago City College, Herzl Branch

JOSEPH LANDIN, University of Notre Dame
E. P. LANE, University of Chicago
LUISE LANGE, Chicago City College, Wilson Branch
R. E. LANGER, University of Wisconsin
H. D. LARSEN, University of New Mexico
GILLIE A. LAREW, Randolph-Macon Woman's College
J. W. LASLEY, JR., University of North Carolina
J. H. LEWIS, New York, N. Y.
LEO LIOLIOS, St. Louis University
Z. L. LOFLIN, Southwestern Louisiana Institute
A. T. LONSETH, Northwestern University
R. C. LUIPPOLD, University of Buffalo

C. C. MACDUFFEE, University of Wisconsin
MORRIS MARDEN, University of Wisconsin in Milwaukee
ELLA MARTH, Harris Teachers College
W. T. MARTIN, Syracuse University
SISTER MARY FELICE, Mt. Mary College
J. R. MAYOR, Southern Illinois Normal University
W. C. MCDANIEL, Southern Illinois Normal University
E. J. MCSHANE, University of Virginia
KARL MENGER, University of Notre Dame
E. J. MICKLE, Ohio State University
A. N. MILGRAM, University of Notre Dame
D. D. MILLER, U. S. Navy
H. J. MISER, Ohio State University
W. L. MISER, Vanderbilt University
E. J. MOULTON, Northwestern University
J. R. MUSSELMAN, Western Reserve University

J. L. NAGLE, Chicago
C. V. NEWSOM, Oberlin College
MARY W. NEWSON, Eureka College
IVAN NIVEN, Purdue University
E. P. NORTHROP, University of Chicago
F. S. NOWLAN, University of British Columbia

SISTER JEANETTE OBRIST, Mount St. Scholastica College
J. M. H. OLMSTED, University of Minnesota
ISAAC OPATOWSKI, University of Chicago

P. M. PEPPER, University of Notre Dame
L. C. PLANT, Michigan State College
J. E. POWELL, Michigan State College
G. B. PRICE, University of Kansas
A. L. PUTNAM, Chicago

TIBOR RADÓ, Ohio State University
P. H. RAKER, General Motors Institute
J. F. RANDOLPH, Oberlin College
W. R. RANSOM, Tufts College
RUTH B. RASMUSEN, Chicago City College, Wilson Branch
C. H. RAWLINS, JR., U. S. Naval Academy
C. B. READ, University of Wichita
MINA S. REES, Hunter College
F. A. REIBER, Chicago
W. T. REID, Northwestern University
HAIM REINGOLD, Illinois Institute of Technology
G. DE B. ROBINSON, University of Toronto
W. H. ROEVER, Washington University
R. E. ROOT, U. S. Naval Academy
G. F. ROSE, University of Wisconsin
A. E. ROSS, St. Louis University
J. B. ROSSER, Cornell University

HANS SAMELSON, Syracuse University
R. G. SANGER, University of Chicago
J. B. SCARBOROUGH, U. S. Naval Academy
SAMUEL SCHECTER, Indiana University
E. W. SCHREIBER, Western Illinois State Teachers College
M. E. SHANKS, University of Missouri
M. F. SMILEY, Lehigh University
C. V. L. SMITH, U. S. Naval Academy
G. W. SMITH, University of Kansas
H. L. SMITH, Louisiana State University
ABRAHAM SPITZBART, University of Wisconsin
C. E. SPRINGER, University of Oklahoma
R. C. STEPHENS, Knox College
B. M. STEWART, Michigan State College
M. H. STONE, Harvard University
E. B. STOUFFER, University of Kansas
W. T. STRATTON, Kansas State College
J. S. STUBBE, University of Illinois
ARTHUR SVOBODA, DePaul University
A. G. SWANSON, General Motors Institute
J. L. SYNGE, Ohio State University

H. P. THIELMAN, Iowa State College
J. M. THOMAS, Duke University
H. C. TRIMBLE, Iowa State Teachers College
A. W. TUCKER, Princeton University
H. L. TURRITTIN, University of Minnesota

E. P. VANCE, Oberlin College
G. B. VAN SCHAACK, U. S. Navy
E. C. VARNUM, Rockford, Illinois
J. I. VASS, University of Wisconsin in Milwaukee
H. E. VAUGHAN, University of Illinois

K. W. WEGNER, Carleton College
E. T. WELMERS, Michigan State College
MARY K. WERKMAN, Parker High School, Chicago
M. E. WESCOTT, Northwestern University
P. M. WHITMAN, Los Alamos, New Mexico
W. M. WHYBURN, Texas Technological College
L. R. WILCOX, Illinois Institute of Technology
R. L. WILDER, University of Michigan
F. B. WILEY, Denison University
S. S. WILKS, Princeton University
R. S. WOLFE, Northwestern University
C. L. WOODS, Emmanuel Missionary College
F. L. WREN, Peabody College

J. H. ZANT, Oklahoma A. and M. College

The meetings were held in the auditorium of the Museum of Science and Industry in Jackson Park at 57th Street and South Shore Drive, and lunches and dinners were available at the cafeteria in the Museum.

The sessions of the American Mathematical Society were held on Friday and in the morning and afternoon on Saturday. On Friday at 7:30 P.M. the Josiah Willard Gibbs Lecture on "Physics and the wave equation" was given by Professor J. C. Slater; and on Saturday at 11:00 A.M. Professor S. M. Ulam gave an address "On the stability of solutions of functional equations." The twenty-seventh Colloquium of the Society consisted of four lectures on Friday and Saturday by Professor Tibor Radó on "Length and area."

The Mathematical Association held its sessions on Saturday at 7:30 P.M. and on Sunday at 9:30 A.M. The program was prepared by a committee consisting of R. L. Jeffery, chairman; S. S. Cairns, and H. P. Thielman.

FIRST SESSION OF THE ASSOCIATION

"Introduction of vector algebra and vector analysis in elementary mathematics courses," by Professor M. R. Hestenes, University of Chicago.

"Quaternions and rotations," by Professor H. S. M. Coxeter, University of Toronto.

SECOND SESSION OF THE ASSOCIATION

"Rehabilitation of graduate work," by Professor J. R. Kline, University of Pennsylvania.

"Graduate training in statistics," by Professor W. G. Cochran, Iowa State College.

"The formulation of statistical tests in terms of admissible decisions and acceptable risks," by Professor Churchill Eisenhart, University of Wisconsin.

MEETING OF THE BOARD OF GOVERNORS

The Board met Saturday at 10:00 A.M. and again at 2:15 P.M. in the Windermere Hotel. Nineteen members of the Board were present, including nine Regional Governors.

The following twenty-nine persons were elected to membership on applications duly certified:

R. V. ANDREE, Ph.M. (Wisconsin) Grad. Teaching Asst., Univ. of Wisconsin, Madison, Wis.

W. T. ARGUE, A.B. (Tulsa) Chief Computer, Seismograph Service Corp., Tulsa 1, Okla.

SISTER M. RITA CECILE BOYLE, M.S. (Michigan) Asst. Prof., Barry Coll., Miami 38, Fla.

W. H. CARNAHAN, A.M. (Indiana) Asst. Prof., Purdue Univ., Lafayette, Ind.

J. F. DALY, Ph.D. (Princeton) Lt. Comdr., U.S.N.R. Bureau of Ships, Navy Dept., Washington, D. C.

T. C. DOYLE, Ph.D. (Princeton) Asst. Prof., Dartmouth Coll., Hanover, N. H.

SAMUEL EILENBERG, Ph.D. (Warsaw) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.

DANIEL FINKEL, B.S. (C.C.N.Y.) Instr., Amherst Coll., Amherst, Mass.

SISTER MARY SERAPHIM GIBBONS, A.M. (Minnesota) Coll. of St. Catherine, St. Paul 1, Minn.

A. M. GLEASON, B.S. (Yale) Lt., U.S.N.R.

C. H. HOLTON, A.M. (Harvard) Asst. Prof., Georgia School of Tech., Atlanta, Ga.

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The Secretary reported the deaths of the following members of the Association:

W. A. Bratton, Emeritus President, Whitman College. (June 1945)

P. J. da Cunha, Professor of mathematics, University of Lisbon. (February 4, 1945)

H. A. DoBell, Professor of mathematics, New York State College for Teachers. (December 8, 1944)

N. Durairajan, Executive Engineer, Mylapore, Madras. (July 15, 1945)

R. R. Fleet, Professor of mathematics and astronomy, Central College, Fayette, Missouri. (December 1, 1944)

J. L. Gibson, Dean and Professor Emeritus of mathematics, University of Utah. (February 10, 1945)

T. F. Holgate, Professor Emeritus of mathematics, Northwestern University. (April 10, 1945)

K. S. K. Iyengar, Professor of mathematics, Central College, Bangalore, India. (June 23, 1944)

M. M. S. Moriarty, Emeritus, High School, Holyoke, Massachusetts. (April 30, 1944)

W. P. Ott, Head of department of mathematics, University of Alabama. (December 25, 1944)

F. E. Riley, Jr., Teacher, High School, Phoenix, Arizona. (Killed in action October 22, 1944)

C. H. Rowe, Professor of mathematics, University of Dublin. (November 1944)

S. T. Sanders, Jr., Professor of mathematics, Southwestern Louisiana Institute. (April 10, 1945)

G. T. Sellew, Professor Emeritus of mathematics, Knox College. (December 27, 1944)

J. A. Swenson, Associate in mathematics, Teachers College, Columbia University. (May 2, 1945)

J. D. Tamarkin, Professor of mathematics, Brown University. (November 1945)

A. L. Underhill, Associate professor of mathematics, University of Minnesota. (January 18, 1945)

P. H. Underwood, Teacher, retired, High School, Houston, Texas. (December 14, 1944)

Willis Whited, Retired, Harrisburg, Pennsylvania. (April 28, 1945)

N. R. Wilson, Professor of mathematics, University of Manitoba. (December 27, 1944)

Among the items of business transacted by the Board were the following:

The By-laws of the new Pacific Northwest Section were presented and approved by vote of the Board.

Professor Sophia Levy McDonald of the University of California was appointed by the Board as Regional Governor for Region 14, to fill the vacancy caused by the change of residence of the recently elected Governor from that region.

The Board recommended two changes in the By-laws of the Association to be voted upon by the Association at a business meeting in 1946:

1. Section 8g to be amended by replacing the words "by the Board" at the end of the first sentence by the words "by the President subject to the approval of the Board"; and to replace the word "Board" in the sixth and eighth lines of the section by the words "Nominating Committee."

2. Amendments to various sections of the By-laws to do away with our present regions and provide for the election of a Governor from each Section of the Association.

Since there are no Sections of the Association in New England or in eastern Canada, it was suggested that if the second proposed change in the By-laws is adopted, special arrangements might be made to have Governors elected from these areas.

The Board voted that the Association become a "constituent member" of the American Council on Education.

The President was authorized to appoint a committee to study the mathematical curriculum at the Junior College level, this committee to work in cooperation with the Committee for Coordination of Studies in Mathematical Education.

The President was also authorized to appoint a committee to study the use of motion picture films, models, and other similar aids in the teaching of mathematics, and to report back to the Board on the matter.

On the recommendation of a special committee that had been appointed to consider the question, it was voted that the Association should not take over the publication of the National Mathematics Magazine.

The Board voted approval of the final report of the War Policy Committee to the Rockefeller Foundation. Also, in accordance with similar action already taken by the Council of the American Mathematical Society, it was voted to discharge this committee, which had been a joint-committee of the Society and the Association.

A proposal from the American Mathematical Society for the formation of a Mathematical Policy Committee was discussed but no action was taken on the matter at this time.

The report of the Committee on Universal Military Service in Peace Time, a sub-committee of the War Policy Committee, was approved.

The Committee on Slaught Memorial Papers reported that the first of these papers would soon be ready for publication. The papers will appear as supplements to the MONTHLY and be sent free of charge to all members of the Association and subscribers to the MONTHLY. The Board voted that no honorarium would be paid to the authors of the papers, but that each author should receive fifty copies of his paper.

The resignation of Professor Saunders Mac Lane as chairman of the Committee on Carus Monographs was reluctantly accepted by the Board, with thanks to him for the service he had given to the work of this committee. The President was authorized to appoint a new chairman for the committee. In line with a suggestion from Professor Mac Lane the Board voted that members of the Committee on Carus Monographs should be appointed for periods of six years, one new member each year, and the Secretary was empowered to institute this rotation among the present members of the committee.

A communication from the George Banta Publishing Company indicated that it would be necessary to draw up a new contract for the publication of the MONTHLY involving an increase of something like fifteen percent in cost. The Executive Committee was authorized to make such a new contract after conference with representatives of the Banta Company.

On nomination of the Editor-in-Chief the Board elected the following associate editors of the MONTHLY for the year 1946:

E. F. BECKENBACH
E. T. BELL
L. M. BLUMENTHAL
N. B. CONKWRIGHT
H. S. M. COXETER
OTTO DUNKEL
H. P. EVANS
HOWARD EVES
B. F. FINKEL
J. S. FRAME
ORRIN FRINK, JR.
MARJORIE GROVES
B. W. JONES
C. V. NEWSOM
HAIM REINGOLD
MARIE J. WEISS

The Board elected Professor J. F. Randolph of Oberlin College as a member of the Finance Committee for a term of four years to succeed Professor D. R. Curtiss.

Annual Business Meeting of the Association

The annual business meeting of the Association was held on Sunday at 9:30 A.M., President MacDuffee presiding.

The results of the election of officers were announced as follows:

First Vice-President for the term 1946–47: W. L. Ayres, Purdue University.

Governors at Large for the term 1946–48: Deane Montgomery, Smith College; C. V. Newsom, Oberlin College.

Resolutions were adopted expressing the thanks of the Association to Major L. R. Lohr, the Director of the Museum of Science and Industry, and his staff for their hospitality and courtesy in extending to the visiting members the facilities of the Museum; and to Professor R. G. Sanger and the other members of his Committee on Arrangements for their thoughtfulness and efficiency in planning for the comfort and convenience of those in attendance.

W. B. Carver, *Secretary-Treasurer*

DECEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday December 1, 1945. Professor C. A. Nelson, Chairman of the Section, presided at the morning and afternoon sessions.

There were forty-eight present, including the following twenty-eight members of the Association: C. B. Allendoerfer, Laura M. Ashbaugh, L. E. Boyer, W. B. Campbell, P. A. Caris, J. E. Davis, F. L. Dennis, Arnold Dresden, Michael Golomb, W. H. Gottschalk, C. E. Heilman, Aida Kalish, J. R. Kline, D. H. Lehmer, F. L. Manning, Clifford Marburger, D. L. McDonough, Lillian Moore, W. R. Murray, C. A. Nelson, J. C. Oxtoby, C. J. Rees, I. J. Schoenberg, G. L. Walker, A. D. Wallace, R. M. Walter, Jean B. Walton, Anna Pell Wheeler.

At the business meeting the following officers were elected for the coming year: Chairman, P. A. Caris, University of Pennsylvania; Secretary, W. H. Gottschalk, University of Pennsylvania. The program committee for the next meeting will be J. A. Clarkson, University of Pennsylvania (Chairman), G. C. Webber, University of Delaware, and W. R. Murray, Franklin and Marshall College. The next meeting will be held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 30, 1946.

The program consisted of the following papers:

1. *Homotopy groups*, by Professor R. H. Fox, Princeton University, introduced by Professor J. C. Oxtoby.

The homotopy groups $\pi_n(Y)$ were defined, and some of the difficulties involved in the unsolved problem of finding an algorithm for their calculation were discussed.

2. *Some graphical methods in the theory of numbers*, by Professor D. H. Lehmer, University of California and Aberdeen Proving Ground.

The speaker discussed elementary graphical processes for establishing arithmetical theorems, and their corresponding analytical identities, concerning compositions and partitions of numbers into parts which are unrestricted or which are restricted in various ways.

3. *Some unsolved problems in the theory of trigonometric series*, by Professor Antoni Zygmund, University of Pennsylvania, introduced by Professor J. C. Oxtoby.

Professor Zygmund remarked that the history of mathematics shows how beneficial the development of the theory of trigonometric series was for the clarification of various general notions of mathematical analysis. The notion of function, that of the definite integral, the theory of sets of points, to mention a few examples only, were developed in close connection with the theory of trigonometric series. The hope was expressed that the still unsolved problems of that theory will considerably influence other branches of analysis. The most important, and still open, problems of trigonometric series are: (1) Is there a continuous function whose Fourier series diverges everywhere? (2) What is the structure of the "sets of uniqueness"?

W. H. GOTTSCHALK, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Normal, May 10–11, 1946
INDIANA
IOWA, Grinnell, April 19–20, 1946
KANSAS, Emporia, April 13, 1946
KENTUCKY, Louisville, April 27, 1946
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., May 4, 1946
MICHIGAN, Ann Arbor, April 13, 1946
MINNESOTA
MISSOURI
NEBRASKA, Omaha, May 4, 1946
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, April 4, 1946
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN, Boulder, Colo., April 19–20, 1946
SOUTHEASTERN, Raleigh, N. C., April 19–20, 1946
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 4, 1946

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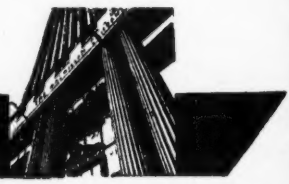

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